Goal

We let M_λ denote the set of all tabloids corresponding to a Young diagram λ. If $\{t\}$ is a fixed tabloid corresponding to λ, the isotropy group of $\{t\}$ is clearly isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_p}$, the subgroup which permutes elements within each row of the diagram. Since S_n acts transitively on M_λ, we see that

$$\# M_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_p!}.$$

Since S_n acts on the set M_λ, we get a representation of S_n on $\mathscr{F}(M_\lambda)$..

We wish to prove the following: to each λ there corresponds a unique 'new' irreducible subrepresentation F_λ of $\mathscr{F}(M_\lambda)$. The space $\mathscr{F}(M_\lambda)$ decomposes into a direct sum of irreducible subrepresentations isomorphic to certain of the F_μ *with* $\mu \geqslant \lambda$ (and these may occur with multiplicity) together with the one unique new subrepresentation F_λ. Thus each Young diagram determines an irreducible representation of S_n.

Young tableaux

By a Young *tableau* corresponding to λ we mean an assignment of the numbers $\{1,\ldots,n\}$ to each of the boxes of λ, one number to each box. In a tableau, the order in each row matters. Thus

3	5	2
1	7	
4		
6		

is a $(3,2,1,1)$ tableau. Each tableau gives rise to a tabloid, by letting the entries in the first row belong to the first set, the entries of the second row correspond to the second set, etc. Two different tableaux, which differ by a permutation of the entries

of their rows, give rise to the same tabloid. If t is a tableau, the corresponding tabloid will be denoted by $\{t\}$. Thus if t is the above tableau, then $\{t\} = \{3,5,2\}\{1,7\}\{4\}\{6\}$.

The column group of a tableau.

Let t be a tableau. Let C_t denote the subgroup of S_n

We now describe the F_λ. Let t be a tableau. Let C_t denote the subgroup of S_n consisting of those π which permute the numbers in the various columns of t among themselves. Thus, if

$$t = \begin{array}{|c|c|c|} \hline 3 & 5 & 2 \\ \hline 1 & 7 & \\ \cline{1-2} 4 & & \\ \cline{1-1} 6 & & \\ \cline{1-1} \end{array}$$

then

$$C_t = S_{\{3,1,4,6\}} \times S_{\{5,7\}}$$

where $S_{\{3,1,4,6\}}$ are the permutations of $\{3,1,4,6\}$, etc.

The e_t.

Let t be a tableau of shape □. Recall that C_t denotes the column group of t. We shall associate to t an element of $\mathcal{F}(M_{□})$ by defining

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}}.$$

Notice that the e_t depends on t and not just the tabloid $\{t\}$

Since $\sigma\delta_{\{t\}} = \delta_{\sigma\{t\}}$, we have

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\sigma\pi\{t\}}.$$

We can write this as

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sign}(\sigma\pi\sigma^{-1})\delta_{\sigma\pi\sigma^{-1}\sigma\{t\}}.$$

But $C_{\sigma t} = \sigma C_t \sigma^{-1}$ so we can write the last sum as

$$\sum_{\rho \in C_{\sigma t}} \operatorname{sgn}(\rho)\delta_{\rho\{\sigma t\}} = e_{\sigma t}.$$

We conclude that

$$\sigma e_t = e_{\sigma t}.$$

The Specht modules F_{λ}.

Since $\sigma e_t = e_{\sigma t}$ we see that the space spanned by the e_t is invariant under S_n.

We define this space to be

F_{λ}. Thus

$$F_{\lambda} = \{\text{linear span of all the } e_t\}$$

as t ranges over all tableaux of shape λ.

Example, the sign representation.

if $\lambda = (1, 1, \ldots, 1)$, then $C_t = S_n$ and, up to sign, there is only

one e_t and it is

$$e_t = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\delta_{\pi t}$$

where we may take

$$t = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \cdot \\ \hline \cdot \\ \hline \cdot \\ \hline n \\ \hline \end{array} \, .$$

The representation in this case is the one-dimensional sign representation. Also in this case $S_{\{t\}} = \{e\}$, so we may identify $M = S_n/S_{\{t\}}$ with S_n and thus $\mathscr{F}(M)$ with $\mathscr{F}(S_n)$. We know that the regular representation contains any irreducible representation with multiplicity equal to its dimension and hence contains $F_{(1,\ldots,1)}$ once. Also $(1, \ldots, 1)$ is the last diagram on our list. This supports the contention of the theorem.

Lemmas about tableaux,1.

(1) Let λ and μ be diagrams and let t be a λ tableau and s be a μ tableau. Suppose that for every i, the numbers from the ith row of s belong to different columns of t. Then $\lambda \geqslant \mu$.

Proof The numbers in the first row of s all lie in different columns of t. Hence, λ has at least μ_1 columns (i.e. $\lambda_1 \geqslant \mu_1$).

We can apply an element of C_t to t so as to arrange that all the elements of the first row of s lie in the first row of t. There may be some extra boxes in this row. Since all the elements of the second row of s lie in different columns of t, we can apply an element of C_t to t so as to arrange that all the elements of the first two rows of s lie in first two rows of t, implying that $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, the hypothesis implies that we can apply an element of C_t to t so as to arrange that all the elements of the first i rows of s Lie in the first I rows of t proving that $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$.

Lemmas about tableaux, 2.

(2) With the same notations as in (1), suppose that

$$\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} \neq 0.$$

Then $\lambda \geqslant \mu$, and if $\lambda = \mu$ then

$$\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} = \pm e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\sigma\pi)\delta_{\pi\{t\}},$$

where $\sigma \in C_t$ and $\{s\} = \sigma\{t\}$.

Proof Let

$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$$

as an operator, so that, for example,

$$A_t\delta_{\{t\}} = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}} = e_t.$$

Lemma 2, continued.

Thus we can rewrite (2) as saying that

$$A_t\delta_{\{s\}} = 0 \quad \text{if} \quad \mu \not\trianglelefteq \lambda$$

and if $\mu = \lambda$ then

$$A_t\delta_{\{s\}} = \begin{matrix} 0 & \text{if } \{s\} \neq \sigma\{t\} \text{ for some } \sigma \in C_t \\ \operatorname{sgn}(\sigma)e_t & \text{if } \{s\} = \sigma\{t\} \text{ for } \sigma \in C_t. \end{matrix}$$

Suppose that

$$A_t\delta_{\{s\}} \neq 0.$$

Under this hypothesis, we claim that two numbers x and y which lie in the same row of s cannot lie in the same column of t. For to say that x and y lie in the same row of s implies that $(xy)\{s\} = \{s\}$. If x and y lie in the same column of t, then $(xy) \in C_t$ and, since $\operatorname{sgn}(xy) = -1$, we could have

$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi = \sum_{\pi \in C_t} \operatorname{sgn}(\pi \cdot (xy))\pi \cdot (xy) = -\sum \operatorname{sgn}(\pi)\pi(xy) = -A_t(xy).$$

Thus

$$A_t\delta_{\{s\}} = -A_t(xy)\delta_{\{s\}} = -A_t\delta_{(xy)\{s\}} = -A_t\delta_{\{s\}}$$

contradicting the assumption that $A_t\delta_{\{s\}} \neq 0$.

Lemma 2, concluded.

This implies that $\lambda \geqslant \mu$. If $\lambda = \mu$, all numbers in the first row of s occur in different columns of t. So we can find some $\pi \in C_t$ such that πt has the same first row as s. All elements of the second row of s occur in different columns of πt and below the first row. So we can find a $\pi' \in C_{\pi t} = C_t$, leaving the numbers in the first row of s fixed with $\pi' \pi t$ having the same first two rows, as s, etc. This shows that $\{s\} = \{\sigma t\}$ for some $\sigma \in C_t$. But then $A_t \delta_{\{s\}} = \operatorname{sgn}(\sigma) e_t$. Thus we have proved (2). We conclude that for any $\{s\}$ whatsoever in M_λ, we have

$$A_t \delta_{\{s\}} = \begin{array}{lll} e_t & \text{if } \{s\} = \sigma\{t\} & \operatorname{sgn}(\sigma) = 1 \\ 0 & \text{if } \{s\} \neq \sigma\{t\} & \\ -e_t & \text{if } \{s\} = \sigma\{t\} & \operatorname{sgn}(\sigma) = -1. \end{array}$$

Lemmas about tableaux, 3.

We have shown that

for any $\{s\}$ whatsoever in M_λ, we have

$$A_t\delta_{\{s\}} = \begin{array}{lll} e_t & \text{if} \quad \{s\} = \sigma\{t\} & \operatorname{sgn}(\sigma) = 1 \\ 0 & \text{if} \quad \{s\} \neq \sigma\{t\} & \\ -e_t & \text{if} \quad \{s\} = \sigma\{t\} & \operatorname{sgn}(\sigma) = -1. \end{array}$$

Now every $f \in \mathscr{F}(M_\lambda)$ is a linear combination of the $\delta_{\{s\}}$ as $\{s\}$ ranges over the λ tabloids. Hence (3).

(3) For any $f \in \mathscr{F}(M_\lambda)$,

$$A_t f = c_f e_t$$

where c_f is a scalar, i.e. $A_t f$ is a multiple of e_t for any f.

Putting a scalar product on $\mathcal{F}$(M□) .

Let us put a scalar product (,) on $\mathcal{F}(M_\lambda)$ by taking the $\delta_{\{t\}}$ as an orthonormal basis. This is clearly S_n invariant. Now for any u, $v \in \mathcal{F}(M_\lambda)$,

$$\begin{aligned}
(A_t u, v) &= \sum_{\pi \in C_t} (\operatorname{sgn}(\pi)\pi u, v) \\
&= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi^{-1})\pi^{-1} v), \text{ since } \operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^{-1}) \\
&= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi)\pi v) \\
&= (u, A_t v).
\end{aligned}$$

In other words, the operator A is self-adjoint relative to (,).

Either or.

(4) Let U be an invariant subspace of $\mathscr{F}(M_\lambda)$. Then either $U \supset F_\lambda$ or $U \subset F_\lambda^\perp$. In particular, F_λ is irreducible.

Proof Let $u \in U$ and let t be a λ tableau. Then $A_t u$ is a multiple of e_t. If for some t and u this multiple is not zero, then $A_t u \in U$ and $A_t u = c_u e_t$, and since F_λ is generated by the σe_t as $\sigma \in S_n$, we see that $F_\lambda \subset U$. If these multiples are zero for all t and u, then $0 = (A_t u, \delta_{\{t\}}) = (u, A_t \delta_{\{t\}}) = (u, e_t)$ for all u and t, so $U \subset F_\lambda^\perp$.

(5) Let $T: \mathscr{F}(M_\lambda) \to \mathscr{F}(M_\mu)$ be any element of $\mathrm{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$. Suppose that $F_\lambda \not\subset \ker T$. Then $\lambda \geqslant \mu$. If $\lambda = \mu$, then the restriction of T to F_λ is a scalar multiple of the identity.

Proof By (4), $\ker T \subset F_\lambda^\perp$. Let t be any λ tableau. Then $0 \neq Te_t = TA_t \delta_{\{t\}} = A_t T \delta_{\{t\}}$. But $T\delta_{\{t\}} \in \mathscr{F}(M_\mu)$ is some combination of $\delta_{\{s\}}$ for μ tabloids $\{s\}$ and $A_t \delta_{\{s\}} = 0$ unless $\lambda \geqslant \mu$. The second part follows from Schur's lemma and (2), since $A_t F_\lambda(M_\lambda) \subset F_\lambda$.

Conclusion of proof.

(6) $\mathrm{Hom}_{S_n}(F_\lambda, F_\mu) = 0$ unless $\lambda = \mu$. In particular, since the number of diagrams = the number of partitions = the number of conjugacy classes of S_n, the F_λ are exactly all the irreducible representations of S_n.

Proof Any $T \in \mathrm{Hom}_{S_n}(F_\lambda, F_\mu)$ can be extended to an element of $\mathrm{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$ by setting it equal to zero on $F_\lambda^\perp$. By (5) this shows that if $T \neq 0$, then $\lambda \geqslant \mu$. Since F_λ and F_μ are irreducible, by Schur's lemma, if $T \neq 0$ then T is invertible and working with T^{-1} shows that $\mu \geqslant \lambda$, hence $\lambda = \mu$.

Standard tableaux.

The elements e_t , as t ranges over all tableaux of shape □ span $F_{□}$, but they are far from being independent. However the following fact is true: A tableau is called **standard** if it increases as we move to the right along rows or down along columns. Then the elements e_t , as t ranges over all standard tableaux of shape □ form a basis of $F_{□}$. We may prove this fact later.