Representation theory of the symmetric groups

Construction of all the irreducible representations.

The conjugacy classes

- Every element of S_n can be written as a product of (disjoint) cycles: For example (1326)(45)(7)(8).
- If $s \in S_n$ is written in cycle form, and t is some other element of S_n , then tst^{-1} is obtained from s by replacing each integer i in the cycle form of s by t(i).
- Conversely, if s_1 and s_2 have the same cycle structure so that there is a permutation t relating the entry i in s_1 to t(i) in s_2 then $s_2 = ts_1t^{-1}$.
- A conjugacy class in S_n is thus determined by $[\square_1,\dots \square_n]$ where $\square_1$ is the number of one cycles, $\square_2$ is the number of two cycles etc.

The number of elements in a conjugacy class.

The v's are constrained by

$$v_1 + 2v_2 + \cdots + nv_n = n.$$

The number of elements in a conjugacy class is given by $\#S_n/\#H$, where H is the isotropy subgroup of some element s in the conjugacy class, i.e. $H = \{t \mid tst^{-1} = s\}$. Suppose s has the cycle structure $[v_1, \ldots, v_n]$. Then t cannot interchange entries coming from cycles of different length. Within the set of cycles of a fixed length, t can act as a cyclic permutation within each cycle and can permute cycles as a whole. Thus,

considering the cycles of different length independently, we see that

$$\#H = 1^{v_1}v_1!2^{v_2}v_2!3^{v_3}v_3!\cdots n^{v_n}v_n!$$

where. for example, 3^{v_3} is present because there are three cyclic permutations within each of the v_3 three-cycles and $v_3!$ is present because there are $v_3!$ permutations of these three-cycles among themselves. Thus,

$$\begin{array}{l}\text{the number of elements}\\ \text{in the conjugacy class}\\ \text{given by } [v_1, \ldots, v_n]\end{array} = \frac{n!}{1^{v_1}v_1!2^{v_2}v_2!\cdots n^{v_n}v_n!}.$$

Example: the conjugacy classes of S_4 .

$$\#\{e\} = \#[4,0,0,0] = \frac{4!}{4!} = 1$$

$$\#\{(a,b)\} = \#[2,1,0,0] = \frac{4!}{2\cdot 2!} = 6$$

$$\#\{(a,b)(c,d)\} = \#[0,2,0,0] = \frac{4!}{2^2\cdot 2!} = 3$$

$$\#\{(abc)\} = \#[1,0,1,0] = \frac{4!}{3} = 8$$

$$\#\{(abcd)\} = \#[0,0,0,1] = \frac{4!}{4} = 6.$$

t

Partitions

Set

$$\begin{aligned}
\lambda_1 &= v_1 + v_2 + \cdots + v_n \\
\lambda_2 &= v_2 + v_3 + \cdots + v_n \\
&\vdots \\
\lambda_n &= v_n
\end{aligned}$$

Thus $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_n$, and it follows from

$$v_1 + 2v_2 + \cdots + nv_n = n$$

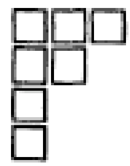
that

$$\lambda_1 + \cdots + \lambda_n = n.$$

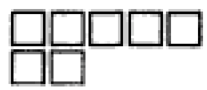
For example, the permutation $(1)(23)(45)(678) \in S_8$ has $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 1$. The set $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called a partition of n. It is conveniently represented by a *Young diagram*.

Young diagrams

We draw the diagram as an array of boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, etc. For example, if $n = 7$ then $\lambda = (3, 2, 1, 1)$ is drawn as



and similarly $(5,2) = (5,2,0,0)$ (we usually drop the zeros) is



Given $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geqslant \lambda_{i+1}$ and $\lambda_1 + \cdots + \lambda_n = n$, we recover v_i by setting

$$v_i = \lambda_i - \lambda_{i+1}.$$

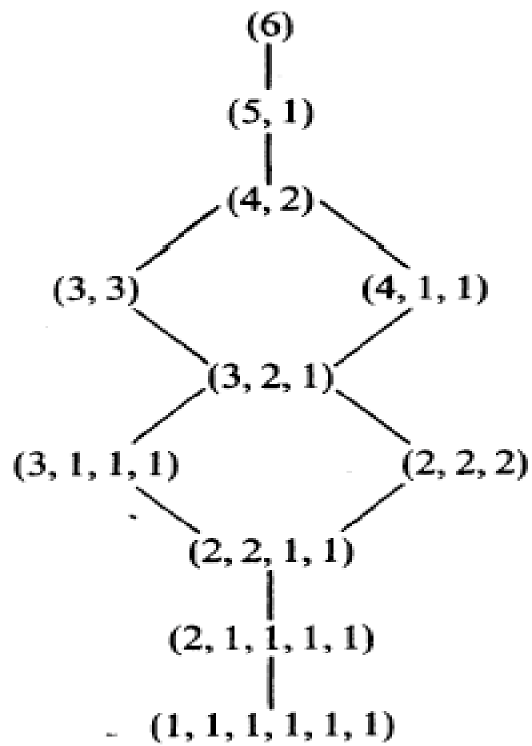
For example, the first diagram corresponds to $v_1 = 1$, $v_2 = 1$, $v_3 = 0$, $v_4 = 1$; the second to $v_1 = 3$, $v_2 = 2$. Clearly $v_1 + 2v_2 + \cdots + nv_n = n$. Thus the number of conjugacy classes of S_n, which is the same as the number of inequivalent irreducible representations of S_n, is the same as the number of Young diagrams. Our task is to attach a distinct irreducible representation of S_n to each diagram. Then we will know that we will hve found all the irreducibles of S_n .

A partial order on the Young diagrams.

From now on we consider a fixed n. We put a partial order on the diagram: saying that $\lambda \geqslant \mu$ if, for all i, the total number of boxes in the first i rows of λ i: less than the total number of boxes in the first i rows of μ; i.e. if

$$\lambda_1 \geqslant \mu_1$$
$$\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 \geqslant \mu_1 + \mu_2 + \mu_3, \text{ etc.}$$

For example, the partial ordering (down is decreasing) for S_6 is given by:



Young tabloids.

By a Young *tabloid* corresponding to the diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$ we mean a decomposition of the set $\{1, \ldots, n\}$ into a union of disjoint sets where the first set contains λ_1 elements, the second set contains λ_2 elements, etc. Thus

$$\{3,5,2\}\{1,7\}\{4\}\{6\} \quad \text{or} \quad \left\{\begin{array}{lll} \boxed{3} & \boxed{5} & \boxed{2} \\ \boxed{1} & \boxed{7} & \\ \boxed{4} & & \\ \boxed{6} & & \end{array}\right\}$$

is a Young tabloid corresponding to the Young diagram $(3, 2, 1, 1)$. The individual subsets are unordered so

$$\{2,3,5\}\{7,1\}\{4\}\{6\} \quad \text{or} \quad \left\{\begin{array}{lll} \boxed{2} & \boxed{3} & \boxed{5} \\ \boxed{7} & \boxed{1} & \\ \boxed{4} & & \\ \boxed{6} & & \end{array}\right\}$$

is the same tabloid.

Young tabloids, continued.

However, the order of the subsets is important,

$$\{3,5,2\}\{1,7\}\{6\}\{4\} \quad \text{or} \quad \left\{ \begin{array}{|c|c|c|} \hline 3 & 5 & 2 \\ \hline 1 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right\}$$

is a different tabloid. We can think of a tabloid as a way of putting the number $\{1,2,\ldots,n\}$ into the boxes of a Young diagram, where the order of numbers within each row does not matter.

We let M_λ denote the set of all tabloids corresponding to a Young diagram λ.

The group S_n acts on M_λ by permuting the elements in the boxes. This action is clearly transitive.

The number of elements in M□ .

If

$\{t\}$ is a fixed tabloid corresponding to λ, the isotropy group of $\{t\}$ is clearly isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_p}$, the subgroup which permutes elements within each row of the diagram. Since S_n acts transitively on M_λ, we see that

$$\# M_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_p!}.$$

The representation of S_n on $\mathcal{F}(M_{\lambda})$.

Since S_n acts on the set M_λ, we get a representation of S_n on $\mathscr{F}(M_\lambda)$..For example, $M_{(n)}$ contains only one element,

$$\{1,\ldots,n\} \quad \text{or} \quad \{\boxed{1}\boxed{2}\boxed{3}\boxed{\cdots}\boxed{n}\}.$$

All permutations carry this tabloid into itself, so the representation of S_n on $\mathscr{F}(M_{(n)})$ is the trivial representation. An element of the set $M_{(n-1,1)}$ is of the form $\{1,\ldots,\hat{k},\ldots,n\} \cup \{k\}$, where the symbol $\hat{k}$ means that k is *missing*. So $M_{(n-1,1)}$ can be identified with the set $\{1,\ldots,n\}$, where $k \in \{1,\ldots,n\}$ corresponds to the missing $\{k\}$. For example, if $n = 3$ there are three tabloids, $\{23\}, \{1\}; \{13\}, \{2\};$ and $\{12\}, \{3\}$, which may be identified with 1, 2, and 3, respectively. We have seen that S_n, acting on $M_{(n-1,1)} \times M_{(n-1,1)}$, has two orbits, so that

$$\mathscr{F}(M_{(n-1,1)}) = \mathbb{C} \oplus F_{(n-1,1)},$$

where $F_{n-1,1}$ is an irreducible space of dimension $n-1$. Notice that the first component, the constant functions, is just $\mathscr{F}(M_{(n)})$.

The representation of S_n on $\mathcal{F}(M_{n-2,2})$.

The set $M_{(n-2,2)}$ $(n > 3)$ can be identified with the space of all two-element subsets of $\{1,\ldots,n\}$, where we look at the entries in the second subset, so that $\{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n\}$, $\{k,l\}$ is identified with $\{k,l\}$. For example, if $n=5$, $M_{(n-2,2)}$ has ten elements. The element $\{3,4,5\}$, $\{1,2\}$ is associated with $\{1,2\}$, the element $\{2,4,5\}$, $\{1,3\}$ with $\{1,3\}$, and so on. A pair of two-element subsets may have either zero, one or two elements in common. Thus, S_n has three orbits when acting on $M_{(n-2,2)} \times M_{(n-2,2)}$, and so $\mathcal{F}(M_{(n-2,2)})$ breaks up into three irreducible components. We claim that two of these components are $\mathcal{F}(M_{(n)})$ and $\mathcal{F}(M_{(n-1,1)})$. Indeed, we now describe a map from $\mathcal{F}(M_{(n-1,1)})$ to $\mathcal{F}(M_{(n-2,2)})$ which commutes with the action of S_n and is injective: we must find a map, T, which goes from functions, f, on $\{1,\ldots,n\}$ to functions on two-element subsets. Take T to be given by

$$(Tf)(\{a,b\}) = f(a) + f(b).$$

It is clear that T commutes with the action of S_n. Also T (constant) = constant and $T\delta_a$ is not a constant (and, in particular, not zero). Thus T is not zero when restricted to each of the irreducible components of $\mathcal{F}(M_{(n-1,1)})$ and hence is injective. Thus

$$\mathcal{F}(M_{(n-2,2)}) = \mathbb{C} + T(F_{(n-1,1)}) + F_{(n-2,2)}$$

$$1 \qquad n-1 \qquad \frac{n(n-3)}{2}.$$

The dimension of $F_{(n-2,2)}$ is obtained by subtracting:

$$\dim F_{(n-2,2)} = \frac{n!}{(n-2)!2!} - n = \frac{n(n-3)}{2}.$$

Goal:

We wish to prove the following: to each λ there corresponds a unique 'new' irreducible subrepresentation F_λ of $\mathscr{F}(M_\lambda)$. The space $\mathscr{F}(M_\lambda)$ decomposes into a direct sum of irreducible subrepresentations isomorphic to certain of the F_μ *with* $\mu \geqslant \lambda$ (and these may occur with multiplicity) together with the one unique new subrepresentation F_λ. Thus each Young diagram determines an irreducible representation of S_n.

Young tableaux.

By a Young *tableau* corresponding to λ we mean an assignment of the numbers $\{1,\ldots,n\}$ to each of the boxes of λ, one number to each box. In a tableau, the order in each row matters. Thus

3	5	2
1	7	
4		
6		

is a $(3,2,1,1)$ tableau. Each tableau gives rise to a tabloid, by letting the entries in the first row belong to the first set, the entries of the second row correspond to the second set, etc. Two different tableaux, which differ by a permutation of the entries of their rows, give rise to the same tabloid. If t is a tableau, the corresponding tabloid will be denoted by $\{t\}$. Thus if t is the above tableau, then $\{t\} = \{3,5,2\}\{1,7\}\{4\}\{6\}$.