The tensor product of representations

Let G and H be groups. Their direct product $G \times H$ consists of all pairs (a, b) with the multiplication law

$$(a, b)(c, d) = (ac, bd).$$

Suppose that (r, U) is a representation of G and (s, V) is a representation of H. We can form the tensor product, $U \otimes V$, of the two vector spaces U and $V^{\dagger}$. Recall from the theory of tensor products that if $A \in \mathrm{Hom}(U, U)$ and $B \in \mathrm{Hom}(V, V)$, then there is a unique transformation $A \otimes B$ on $U \otimes V$ such that

$$(A \otimes B)(\mathbf{u} \otimes \mathbf{v}) = A\mathbf{u} \otimes B\mathbf{v}.$$

Also

$$\mathrm{tr}\,(A \otimes B) = (\mathrm{tr}\,A) \cdot (\mathrm{tr}\,B).$$

Furthermore, if $C \in \mathrm{Hom}(U, U)$ and $D \in \mathrm{Hom}(V, V)$, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

The character of tensor products

This shows that we get a representation $r \otimes s$ of $G \times H$ on $U \otimes V$ by setting

$$(r \otimes s)(a, b) = r(a) \otimes s(b),$$

and that

$$\chi^{r \otimes s}(a, b) = \chi^r(a)\chi^s(b).$$

If $(\ ,\ )_{G \times H}$ denotes the scalar product on $G \times H$, and $\|\ \|_{G \times H}$ the corresponding norm, then

$$\begin{aligned}\|\chi^{r \otimes s}\|^2_{G \times H} &= (\chi^{r \otimes s}, \chi^{r \otimes s})_{G \times H} \\ &= \frac{1}{\#(G \times H)} \sum_{\substack{a \in G \\ b \in H}} \chi^{r \otimes s}(a, b)\overline{\chi^{r \otimes s}(a, b)} \\ &= \frac{1}{\#G} \cdot \frac{1}{\#H} \left(\sum_{a \in G} \chi^r(a)\overline{\chi^r(a)} \right) \left(\sum_{b \in H} \chi^s(b)\overline{\chi^s(b)} \right) \\ &= \|\chi^r\|^2_G \|\chi^s\|^2_H.\end{aligned}$$

In particular, if $\|\chi^r\|^2_G = \|\chi^s\|^2_H = 1$, then $\|\chi^{r \otimes s}\|^2_{G \times H} = 1$. Thus,

If r is an irreducible representation of G and s is an irreducible representation of H, then $r \otimes s$ is an irreducible representation of $G \times H$.

Application to the regular representation

Let r be a representation of G on W. We can construct a representation $\hat{r}$ of G on the dual space W^* by defining

$$\hat{r}(a)l = r(a)^{*-1}l.$$

This is a representation because

$$\begin{aligned} r(ab)^{*-1} &= (r(a)r(b))^{*-1} \\ &= (r(b)^* r(a)^*)^{-1} \\ &= r(a)^{*-1} r(b)^{*-1} \\ &= \hat{r}(a)\hat{r}(b). \end{aligned}$$

We thus get a representation of $G \times G$ on $W \otimes W^*$. If the representation of G on W is irreducible, then so is the representation of $G \times G$ on $W \otimes W^*$.

Application to the regular representation, continued.

We have seen how to attach a function $f^l_{\mathbf{w}}$ on

G to each pair $(\mathbf{w}, l)$ with $\mathbf{w} \in W$ and $l \in W^*$. Since $f^l_{\mathbf{w}}$ depends linearly on $\mathbf{w}$ for l fixed, and linearly on l for $\mathbf{w}$ fixed, we have thus defined a map

$$W \otimes W^* \to \mathcal{F}(G)$$
$$w \otimes l \rightsquigarrow f^l_{\mathbf{w}}.$$

Now the group $G \times G$ acts on G by right and left multiplication:

$$(a, b)c = acb^{-1}$$

and hence we get a corresponding representation ℓ^G on $\mathcal{F}(G)$

$$[\ell^G(a, b)f](c) = f(a^{-1}cb).$$

Application to the regular representation, continued.

Notice that

$$\begin{aligned} f_{r(a)\mathbf{w}}^{\check{r}(b)l}(c) &= \langle r(c)^{-1} r(a)\mathbf{w}, r(b)^{*-1} l\rangle \\ &= \langle r(b)^{-1} r(c)^{-1} r(a)\mathbf{w}, l\rangle \\ &= \langle r(a^{-1}cb)^{-1}\mathbf{w}, l\rangle \\ &= f_{\mathbf{w}}^{l}(a^{-1}cb). \end{aligned}$$

In other words, the map from $W \otimes W^*$ to $\mathscr{F}(G)$ is a morphism for the action of $G \times G$.

Now decompose $\mathscr{F}(G)$ into irreducibles under the action of $G \times G$: For each irreducible representation W_i of G, we know that $W_i \otimes W_i^*$ occurs as an irreducible component under $G \times G$ on $\mathscr{F}(G)$. Under G, the space $W_i \otimes W_i^*$ decomposes into a direct sum of n_i copies of W_i. In particular, no $W_i \otimes W_i^*$ has any component in common with $W_j \otimes W_j^*$ for $i \neq j$. *A fortiori*, $W_i \otimes W_i^*$ and $W_j \otimes W_j^*$ are inequivalent as irreducible representations of $G \times G$. Thus $W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$ occurs as a summand of $\mathscr{F}(G)$, where $W_1, \ldots, W_k$ are all the irreducible representations of G. But the dimension of this summand is $\sum n_i^2 = \dim \mathscr{F}(G)$. Thus

$$\mathscr{F}(G) = W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$$

In the decomposition of $\mathscr{F}(G)$ into irreducibles under GxG.

Each irreducible summand occurs exactly **once** .

of irreducibles = # of conjugacy classes.

In the decomposition $\mathscr{F}(G) = W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$ each summand is irreducible and occurs once. Hence

$$\dim \operatorname{Hom}_{G \times G}(\mathscr{F}(G), \mathscr{F}(G)) = \underbrace{1^2 + \cdots + 1^2}_{k \text{ times}} = k.$$

We know that this dimension must equal the number of orbits of $G \times G$ acting on $G \times G$ by the rule

$$(a, b)(c, d) = (acb^{-1}, adb^{-1}).$$

we can always find an element of the form (e, d) on any orbit. But

$(a, b)(e, d) = (ab^{-1}, adb^{-1})$ will have the same form if $b = a$. Thus (e, d) and (e, ada^{-1}) lie on the same orbit, and hence

> the number of orbits of $G \times G$ on $G \times G$ is equal to the number of conjugacy classes of G.

Thus $\dim \operatorname{Hom}_{G \times G}(\mathscr{F}(G), \mathscr{F}(G)) = k = \#$ of conjugacy classes.

We have proved that

> the number of distinct irreducible representations is equal to the number of conjugacy classes. (6.3)

A second orthogonality relation for characters.

Let C denote the space of functions which are constant on conjugacy classes, and let $\chi_1, \ldots, \chi_k$ be the distinct irreducible characters. We already know that the functions $\chi_i \in C$ are mutually orthogonal and have length one. Since $k = \#$ of conjugacy classes $= \dim C$, they form an orthonormal basis of C. Any $f \in C$ can be expanded in terms of the basis $\chi_1, \ldots, \chi_p$:

$$f = (f, \chi_1)\chi_1 + \cdots + (f, \chi_p)\chi_p.$$

Let us apply this formula to the function f_j, which equals one on the jth conjugacy class and vanishes on all the others. Then $(f, \chi_i) = (\#C_j/\#G)\overline{\chi_i(j)}$, where $\#C_j$ is the number of elements in the jth conjugacy class, C_j, and $\chi_i(j)$ is the value of χ_i on any element of this class. Substituting into the above formula and evaluating at a point in the jth conjugacy class, C_j, we get

$$1 = (\#C_j/\#G)(\chi_1(j)\overline{\chi_1(j)} + \cdots + \chi_p(j)\overline{\chi_p(j)}). \tag{6.4}$$

Evaluating at a different conjugacy class gives

$$0 = \chi_1(k)\overline{\chi_1(j)} + \cdots + \chi_p(k)\overline{\chi_p(j)} \quad \text{if } j \neq k. \tag{6.5}$$

The two orthogonality relations for characters

Let $\chi_1, \ldots, \chi_p$ be the distinct irreducible characters of the group G, and let $C_1, \ldots, C_p$ denote the distinct conjugacy classes. We denote by $\chi_i(j)$ the (constant) value of the character χ_i on any element of the conjugacy class C_j. Then

$$(\chi_i, \chi_k) = (1/\#G) \sum_{a \in G} \chi_i(a)\overline{\chi_k(a)} = (1/\#G) \sum_{j=1}^{p} (\#C_j)\chi_i(j)\overline{\chi_k(j)}.$$

We can thus write the orthogonality relations (4.6) and (4.7) as

$$(\#C_1)\chi_i(1)\overline{\chi_k(1)} + \cdots + (\#C_p)\chi_i(p)\overline{\chi_k(p)} = \begin{cases} \#G & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases}. \tag{7.1}$$

We can write the orthogonality relations (6.4) and (6.5) as

$$(\#C_j)[\chi_1(j)\overline{\chi_1(l)} + \cdots + \chi_p(j)\overline{\chi_p(l)}] = \begin{cases} \#G & \text{if} \quad j = l \\ 0 & j \neq l \end{cases}. \tag{7.2}$$

Character tables

Equations (7.1) and (7.2) can be summarized in the form of a table. We label the columns by the conjugacy classes, indicating, alongside C_j, the number of its elements $\#C_j$. We label the rows by the characters χ_i, and place the value $\chi_i(j)$ in the i, j position. Then (7.1) says that the 'scalar product' of two distinct rows is zero, and of a row with itself is $\#G$, provided that we weight the jth column by $\#C_j$. Similarly, (7.2) says the same thing about the scalar product of the columns, again weighting the columns by $\#C_j$. The table so obtained is called the character table of the group. In a sense, it contains all the information about the representations of the group. The first conjugacy class, C_1, is usually taken to be the one-element conjugacy class $[e]$. Thus, the elements of the first column consist of the values $\chi_i(e) = n_i$, the degree of the ith irreducible representation. The character χ_1 is usually taken to be the trivial representation, so that the entries of the first row are all ones.

The character table of S_3

$6S_3$	$1C_1$	$3C_2$	$2C_3$
χ_1	1	1	1
χ_2	2	0	-1
χ_3	1	-1	1

For $j = l = 2$, and $j = l = 3$ (7.2) says, respectively,

$$3[1^2 + 0^2 + (-1)^2] = 6$$

and

$$2[1^2 + (-1)^2 + 1^2] = 6.$$

Notice that all the entries of the character table for S_3 happen to be integers.

The character table of C_n .

nC_n	$1[e]$	$1[a]$	$1[a^2]$	$\cdots$	$1[a^{n-1}]$
χ_1	1	1	1	$\cdots$	1
χ_2	1	ε	ε^2	$\cdots$	ε^{n-1}
χ_3	1	ε^2	ε^4	$\cdots$	$\varepsilon^{2(n-1)}$
.	.	.	.	$\cdots$	.
.	.	.	.	$\cdots$	.
.	.	.	.	$\cdots$	.
χ_n	1	ε^{n-1}	$\varepsilon^{2(n-1)}$	$\cdots$	$\varepsilon^{(n-1)^2}$

Let a be a generator for C_n, so that the conjugacy classes are the various $[a^{j-1}]$, $j = 1, 2, \ldots,$ n. Let ε be a primitive nth root of unity. Then the characters χ_i, determined by $\chi_i(a) = \varepsilon^{i-1}$, $i = 1, 2, \ldots, n$, are all distinct, and thus give all the characters.

The character table of T.

$12T$	$[e]$	$4[r_3]$	$4[r_3^2]$	$3[r^2]$
χ_1	1	1	1	1
χ_2	1	ε	ε^2	1
χ_3	1	ε^2	ε	1
χ_4	3	0	0	-1

$\varepsilon = \exp 2\pi i/3$

Let us now compute the character table of the group T. This group is of order 12, and has a three-dimensional representation as the symmetries of the tetrahedron. The trace of any rotation through angle ϕ in $\mathbb{R}^3$ (or $\mathbb{C}^3$) is $1 + 2\cos\phi$. Thus, for this three-dimensional representation we have

$$\chi(e) = 3, \quad \chi(R_{120^\circ}) = \chi(R_{240^\circ}) = 0 \quad \text{and} \quad \chi(R_{180^\circ}) = -1.$$

Thus

$$(\#G)\|\chi\|^2 = 9 + 3 \cdot 1 = 12$$

since there are three rotations through 180° and four each through 120° and 240°. We see that this three-dimensional representation is irreducible. Since the sum of squares of the degrees of all irreducible representations is 12, and $3^2 = 9$, there must also be three one-dimensional representations. These can be found as follows: let H be the subgroup of T consisting of the identity and the rotations through 180°. Then H is a normal subgroup and hence any representation of the quotient group, T/H, lifts to a representation of T. But T/H is just the cycle group C_3, which has three one-dimensional representations. Thus the character table of T is given by Table 8.