Math 126 Lecture 5

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Until further notice, all groups will be finite, and all vector spaces will be finite dimensional vector spaces over the complex numbers. A typical group will be denoted by G or H, the identity element by e and vector spaces by V, W,etc.

1 Representations.

A **representation** r of G on V is an action of G on V in which every element acts as a linear transformation. So r is a homomorphism of G into the group $Gl(V)$ of all invertible linear transformations of V. We will sometimes write $r(a)$ for the image of a under r, so

$$r(a)v$$

is the image of $v \in V$ under the action of the linear transformation $r(a)$; or, if r is clear from the context we may write

$$av$$

instead of $r(a)v$.

If we choose a basis of V then we will write the matrix associated to $r(a)$ by this basis as

$$(r_{ij}(a)).$$

The matrices $(r_{ij}(a))$ depend on the choice of basis, and so are determined only up to conjugacy. The fact that r is a representation, so

$$r(e) = I$$

where I is the identity operator, and

$$r(ab) = r(a)r(b)$$

translates into

$$r_{ij}(e) = \delta_{ij} \quad \text{and} \quad r_{ij}(ab) = \sum_k r_{ik}(a)r_{kj}(b).$$

1.1 Equivalence.

Let r and r' be representations of the same group G on vector spaces V and V'. We say that r and r' are **similar** or **equivalent** if there is a bijective (i.e. one to one and onto) linear map $T : V \to V'$ which is a morphism, i.e. such that

$$r'(a)T = Tr(a) \quad \forall \ a \in G.$$

The space of all $T \in \mathrm{Hom}(V, V')$ which satisfy the above equation is denoted by

$$\mathrm{Hom}_G(V, V').$$

1.2 Invariant subspaces.

A subspace of V is **invariant** if $r(a)W \subset W$ for all $a \in G$. Applying $r(a^{-1})$ we see that this is equivalent to $r(a)W = W$ for all $a \in G$. The restriction of $r(a)$ to W is then a representation of G on W called a **subrepresentation** of r.

Clearly V itself and the subspace $\{0\}$ are invariant subspaces. If these are the only irreducible subspaces the representation r is called **irreducible**.

1.3 Examples.

1.3.1 One dimensional representations.

If V is one dimensional, then r is irreducible since there is no room for any subspace to lie strictly between $\{0\}$ and V. A linear transformation on one dimensional space is just multiplication by a scalar. So a one dimensional representation of G is a map

$$\kappa : G \to \mathbb{C}$$

such that

$$\kappa(ab) = \kappa(a)\kappa(b), \qquad \kappa(e) = 1.$$

Since $a^{\#G} = e$, we see that $\kappa(a)$ is root of unity whose order is some divisor of $\#G$.

If $G = C_n$, and let b be a generator of C_n. For each integer k with $0 \leq k < n$ we may define

$$\kappa_k(b) = e^{2\pi ik/n} \quad \text{so} \quad \kappa_k(b^s) = e^{2\pi isk/n}.$$

These are inequivalent irreducible representations of C_n and in fact all of them (as we shall prove), and play a central role in the Fourier transform.

In fact, if G is commutative, all non trivial irreducible representations are one dimensional: Choose $a \in G$. The operator $r(a)$ has at least one eigenvalue, call it $\lambda(a)$. The set W of all $v \in V$ which are eigenvalues of a with eigenvalue $\lambda(a)$ is an invariant subspace. Proof: If $w \in W$ then $r(b)w \in W$ for all $b \in G$ since

$$r(a)r(b)w = r(ab))w = r(ba)w = r(b)r(a)w = r(b)\lambda(a)w = \lambda(a)r(b)w.$$

Since $w =\neq \{0\}$ we must have $W = V$ so $r(a) = \lambda(a)I$. Choose another $c \in G$. Again all $v \in V$ are eigenvectors for $r(c)$. Continuing, we see that every $a \in G$ is represented by a scalar operator $\lambda(a)I$. But then any subspace is invariant, so V must be one dimensional.

1.3.2 Direct sum of two representations.

let r be a representation of G on V and s a representation of G on W. Then $r \oplus s$ is the representation of G on $V \oplus W$ given by $a \mapsto r(a) \oplus s(a)$. In terms of a basis of V and basis of W which are combined to form a basis of $V \oplus W$ the matrix form of $r \oplus s$ is "block diagonal" with (r_{ij}) in the upper left corner and $(s_{k\ell})$ in the lower right corner.

1.3.3 Tensor product of two representations.

Similarly, we define the representation $r \otimes s$ on $V \otimes W$ as being

$$(r \otimes s)(a) = r(a) \otimes s(a).$$

If $e_1, \ldots, e_m$ is a basis of V and $f_1, \ldots, f_n$ is a basis of W then the $e_i \otimes f_k$ form a basis of $V \otimes W$ and the corresponding matrix form of $r \otimes s$ is

$$(r \otimes s)_{ik,j\ell}(a) = r_{ij}(a)s_{k\ell}(a).$$

For example, in the case of the one dimensional representations of C_n discussed above,

$$\kappa_k \otimes \kappa_\ell = \kappa_s \ \text{ where } s \equiv k + \ell \bmod n.$$

2 Unitary representations.

Let $(\ ,\ )$ be a scalar product on V. We say that $(\ ,\ )$ is **invariant** under a representation r of G on V, or that r is **unitary** relative to $(\ ,\ )$, if

$$(r(a)u, r(a)v) = (u, v) \ \ \forall a \in G, \quad u, v, \in V.$$

2.1 Averaging over the group.

If we start with a scalar product $\langle\ ,\ \rangle$ which is not necessarily invariant we can produce an invariant one by defining

$$(u, v) = \frac{1}{\#G} \sum_{b \in G} \langle r(b)u, r(b)v \rangle.$$

This is clearly linear in u, anti-linear in v and $(u.u) > 0$ so is a scalar product. It is invariant since

$$(r(a)u, r(a)v)) = \frac{1}{\#G} \sum_{b \in G} \langle r(b)r(a)u, r(b)r(a)v \rangle = \frac{1}{\#G} \sum_{c \in G} \langle r(c)u, r(c)v \rangle = (u, v)$$

since summing over ab for fixed a is the same as summing over all $c \in G$.

2.2 Maschke's theorem.

This says that if W is an invariant subspace then it has an invariant complement.
Proof: Choose an invariant scalar product. Then the orthogonal complement $W^\perp$ with respect to this scalar product is invariant.

In particular it follows that every representation is equivalent to a finite direct sum of irreducible representations.

3 Schur's lemma.

Let r and s be irreducible representations of G on V and W, and $T \in \mathrm{Hom}_G(V, W)$. Schur's lemma makes two assertions:

$$r \not\sim s \quad \Rightarrow T = 0, \tag{1}$$

and, if

$$r = s \quad (\text{so } V = W) \ \text{ then } T = zI \tag{2}$$

for some scalar z.

Proof of (1): The subspace $\ker T \subset V$ is invariant. so the alternatives are $\ker T = V$ in which case $T = 0$, or $\ker T = \{0\}$, in which case T is one to one. Also Im $(T) \subset W$ is invariant so $= \{0\}$ and so $T = 0$ or $\mathrm{Im} T = W$ is which case T is surjective so an equivalence contrary to hypothesis.

Proof of (2): Apply (1) to $T - cI$ where c is an eigenvalue of T. We must have $T - cI = 0$ or $T = cI$. □

3.1 Averaging over the group.

G acts on $\mathrm{Hom}(V,W)$ by letting $a \in G$ send $Q \in \mathrm{Hom}(V,W)$ into

$$s(a)Qr(a)^{-1}.$$

It is esy to check that this is a representation of G on $\mathrm{Hom}(V,W)$ and that $T \in \mathrm{Hom}(V,W)$ belongs to $\mathrm{Hom}_G(V,W)$ if and only if $T \in \mathrm{Fix}(G)$. Starting with any $Q \in \mathrm{Hom}(V,W)$ the element

$$Q_{av} := \frac{1}{\#G}\sum_{b\in G} s(b)Qr(b)^{-1}$$

belongs to $\mathrm{Fix}(G)$ and so to $\mathrm{Hom}(V,W)$. It follows from (1) that if r and s are irreducible then

$$r \not\sim s \Rightarrow Q_{av} = 0, \tag{3}$$

while if $r = s$ then (2) implies that

$$Q_{av} = cI. \tag{4}$$

3.2 Orthogonality relations for matrix elements.

Choose a basis of V and of W so we get square matrices $(r_{ij}(a))$ and $(s_{k\ell}(a))$. We can also write Q as a (rectangular) matrix q_{ki}. Then Q_{av} has as its matrix entries

$$\frac{1}{\#G}\sum_{a\in G,\ell,k} s_{k\ell}(a)q_{\ell i}r_{ij}(a^{-1}).$$

If $r \not\sim s$ this must vanish for all $q_{\ell i}$ which tells us that

$$r \not\sim s \quad \Rightarrow \frac{1}{\#G}\sum_{a\in G} s_{k\ell}(a)q_{elli}r_{ij}(a^{-1}) = 0 \quad \forall\ i,j,k,\ell. \tag{5}$$

If $r = s$ then (3) implies that

$$\frac{1}{\#G}\sum r_{\ell k}(a)q_{ki}r_{ij}(a^{-1}) = zI$$

and taking the trace of both sides shows that

$$z = \frac{1}{n}\,\mathrm{tr}(Q)$$

where $n = \dim V$. Comparing the coefficients of both sides shows that if r is irreducible then

$$\frac{1}{\#G}\sum_{a\in G} r_{k\ell}(a)r_{ij}(a^{-1}) = \frac{1}{n}\delta_{\ell i}\delta_{kj}, \quad n = \dim V. \tag{6}$$

These equations take on a more pleasant form if we restrict attention to unitary representations and use orthonormal bases so that

$$r_{ij}(a^{-1}) = \overline{r_{ji}(a)}$$

and similarly for s. Let $\mathcal{F}(G)$ denote the vector space of all complex valued functions defined on G and put the scalar product

$$(f, g) := \frac{1}{\#G} \sum_{a \in G} f(a)\overline{g(a)}$$

on $\mathcal{F}(G)$. Then (5) becomes

$$r \not\sim s \quad \Rightarrow (r_{ij}, s_{k\ell}) = 0 \quad \forall \, ijk\ell \tag{7}$$

while (6) becomes

$$(r_{ij}, r_{kl}) = \frac{1}{n}\delta_{ki}\delta_{\ell j} \tag{8}$$

In short, if we regard the matrix entries of irreducible unitary representations as functions on G, then matrix entries from inequivalent representations are orthogonal while different matrix entries from the same irreducible representation are also orthogonal, and each matrix entry has square length $1/n$ where $n = \dim V$.