Math. 126 Problem set 8

Young's rule and the hook formula.

Nov. 19, 2002 due Nov. 26

Contents

1 Young's rule. **1**

2 The Hook formula. **2**

1 Young's rule.

1. Multiply the three rows □□□□, □□, and □□ according to the Pieri formula which is a special case of the Littlewood Richardson rule. What is the coefficient of each diagram that occurs in your answer?

2. For each diagram λ that occurs in your answer to Problem **1**, count the number of (semi-standard) tableaux of shape λ and weight $(4, 2, 2, 0, 0, \dots)$. These numbers should be the same as the coefficients. Why is this so?

3. Prove **Young's rule** which says that for any pair of diagrams λ and μ both with n boxes,

$$\dim \operatorname{Hom}_{S_n}(F^\lambda, M^\mu) = \# \text{ of tableaux of shape } \lambda \text{ ane weight } \mu.$$

A **standard tableau** is one in which no repetitions are allowed. So if λ is a diagram with n boxes, then a standard tableau of shape λ is a way of putting the numbers $1, \dots, n$ into the boxes of λ so that the numbers strictly increase (from left to right) along rows and (from top down) along columns.

4. Prove that

$$\dim F^\lambda = \#\text{of standard tableaux of shape } \lambda.$$

2 The Hook formula.

We want to prove that

$$\# \text{ of standard tableaux of shape } \lambda \quad = \quad \frac{n!}{\prod \text{hook lengths}}.$$

Please go back and review the probabilistic algorithm described in Problem Set **5**. In problem **5** of that problem set, you computed that the probability of ending up with any standard tableau for the diagram $(3, 2)$ was $\frac{1}{5}$. Since all tableaux in this example have the same probability, and since the probabilities must add up to 1, we conclude that there must be exactly five tableaux. So in general, we will have proved the above formula, if we can prove that the probabilistic algorithm when applied to any diagram λ with n boxes yields all standard tableaux with the same probability given by

$$\frac{\prod \text{hook lengths}}{n!}.$$

This formula is clearly true when $n = 1$, so we will prove it by induction on the number of boxes. So we assume that it is true for any diagram with $n-1$ boxes. We know that when applying the algorithm we will end up at some corner c with some probability, call it p_c. Let λ^c denote the diagram (with $(n-1)$ boxes) obtained by removing the corner c from the diagram λ (assumed to have n boxes). We will be done if we can prove that

$$p_c = \frac{\frac{\prod \text{hook lengths of } \lambda}{n!}}{\frac{\prod \text{hook lengths of } \lambda^c}{(n-1)!}}.$$

Notice that there is a lot of cancellation. The factorials almost cancel leaving a factor of $\frac{1}{n}$. Any box of λ which does not lie on either the same row or the same column as c makes the same contribution to numerator and denominator. So the only contributions come from boxes b of λ such that $c \in H_b,\ b \neq c$ where H_b is the hook emanating from b. These hooks have one less box in λ^c than in λ. So if h_b denotes the hook length in λ of H_b we will be done if we can prove that

$$p_c = \frac{1}{n} \prod_{b \in \lambda, c \in H_b, b \neq c} \frac{h_b}{h_b - 1}.$$

We know what the $\frac{1}{n}$ means. This is just the probability at the very start of the algorithm of picking any particular box in λ. So we must understand the product. Let R denote the row containing c and let C denote the column containing c. We can write the product as

$$\prod_{b \in R, b \neq c} \left(1 + \frac{1}{h_b - 1}\right) \prod_{b \in C, b \neq c} \left(1 + \frac{1}{h_b - 1}\right). \tag{1}$$

5. Suppose that the initial choice of a box landed you on the same row as c. In other words, suppose that the initial choice landed you at $b \in R$. In order to end up at c, all successive choices must keep you on the same row. Let $b_1, b_2, \ldots, b_k, b_{k+1} = c$ be the path traversed to get to c where b_1 was the initial choice. What is the probability of making these choices? How does this probability relate to the expansion of the product above? Do the same for the column containing c.

Now to the general case! In order to end up at c, the initial choice had to be to the northwest of c, that is, the initial choice had to at a box whose row was $\leq$ the row of c and whose column was $\leq$ the column of c. Proceeding from that position according to the algorithm will yield a subset I of rows $<$ the row of c, and a subset J of columns $<$ the column of c. Each such pair of subsets corresponds to possible row and column values in a path of the algorithm terminating at c, and also corresponds to a term in the expansion of the product.

For example, consider

$$\begin{array}{ccccccc}
\square & \square & \square & \square & \square & \square & \square \\
\square & \Delta & \square & \square & \bullet & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \bigcirc & \square & \square & \mathbf{c} & & \\
\square & \square & \square & & & & \\
\square & \square & & & & & \\
\square & & & & & &
\end{array} .$$

The position of the corner **c** is $(4, 5)$. Starting at Δ which is $(2, 2)$ there are several ways of ending at **c**. In the picture we have described two possible routes in which $I = \{2\}$ and $J = \{2\}$. Since the hook length of Δ is 10, the probability of of moving from Δ to $\bullet$ is $\frac{1}{9}$ as is the probability of moving from Δ to $\bigcirc$. The hook length of $\bullet$ is 5, and the hook length of $\bigcirc$ is 6. So the combined probability of following a route for which $I = \{2\}$ and $J = \{2\}$ is

$$\frac{1}{9}\left(\frac{1}{4} + \frac{1}{5}\right) = \frac{1}{4} \cdot \frac{1}{5}$$

which is exactly the corresponding term in the expansion of the product (1).

6. Show that every term in the expansion of the product (1) is the sum of the probabilities of routes terminating at c with the pair I, J as the rows and columns of the route. [HInt; use induction on $\#I + \#J$.]