Math 126 Exercise set 5.

Representations of the symmetric group.

Oct.24, 2002,   due Oct. 31.

1. Let χ be a character of an irreducible representation of S_n and $a \in S_n$. Show that $\chi(a)$ is an integer. [Of course, once we know that all irreducible characters take on only integer values, this is then true for all characters, since any character is a sum of irreducibles.]

The **Murnahan-Nakayama rule** which I will describe below gives an algorithm for computing the value of the character associated to any Young diagram at any conjugacy class. But before I describe this rule, I want to remind you of the hook formula for the dimension of the irreducible associated to any Young diagram. The **hook** associated to any position in a Young diagram is the collection of all positions to the right (on the same row) and below (on the same column) including the starting position itself. The **hook length** of any position is the number of boxes in the hook. For example, in the diagram

$$\begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & & & \end{array}$$

the hook lengths are

$$\begin{array}{ccccc} 6 & 5 & 3 & 2 & 1 \\ 2 & 1 & & & \end{array}.$$

The **hook formula** for dimension says that the dimension of the irreducible representation associated to a Young diagram is

$$\frac{n!}{\prod \text{all hook lengths}}.$$

For example, in the diagram pictured above this formula yields

$$\frac{7!}{6 \cdot 5 \cdot 3 \cdot 2 \cdot 2} = 14.$$

We know from class that the dimension of the irreducible associated to the diagram with two rows and two boxes on the second row is

$$\frac{n(n-3)}{2}.$$

For the case $n = 7$ this comes out to 14, so at least we have checked the formula in this special case.

2. Using the hook formula for dimension, calculate all the dimensions of the irreducibles of S_6 and check that the sum of their squares is 720.

Another theorem that we stated without proof in class is that the e_t as t ranges over all standard tableaux of a diagram, form a basis of the irreducible associated to the diagram, where "standard" means that the entries of t increase from left to right along rows and increase down along columns.

3. Find all standard tableaux for the diagram drawn above, and check that there are fourteen of them.

If you believe both of the statements, then you must believe that the number of standard tableaux of a Young diagram is given by the hook formula. I am now going to describe a probabilistic algorithm which is a piece of a proof which I may give later in class for the fact that the number of standard tableaux is given by the hook formula. The algorithm first describes a procedure for deciding where to place the integer n in the diagram, and then by iteration, where to place the integer $n-1$ in the diagram with the position of the n removed etc.

In a standard tableau, the integer n must go into a box which has no boxes beneath it or to its right. Let us call such a position a **corner**. For example, in the diagram drawn above, there are two corners: the rightmost position on the first row and the rightmost position on the second row. The algorithm I am about to describe gives a probability to each of the corners of the diagram. It works as follows:

- Pick a box with probability $1/n$.
- If the box you picked is a corner, stay put.
- If not, then the hook length h of the box is at least two. Move to one of the remaining boxes in the hook, with equal probability, i.e. with probability $1/(h-1)$.
- Continue, accumulating the probabilities.

For example, in the diagram

$$\begin{array}{lll} \square & \square & \square \\ \square & & \end{array}$$

the first stage assigns probabilities

$$\begin{array}{lll} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & & \end{array}.$$

In the second stage, the corners stay put, the second box on the first row must move to the right, and the first box on the first row moves to each of the remaining three boxes with probability $\frac{1}{3}$. So after the second stage the probabilities are

$$\begin{array}{ccc} 0 & \frac{1}{12} & \frac{7}{12} \\ \frac{4}{12} & & \end{array}$$

and after the third and final stage the probabilities are

$$\begin{array}{ccc} 0 & 0 & \frac{8}{12} \\ \frac{4}{12} & & \end{array} .$$

So there is a probability $\frac{2}{3}$ of putting the 4 at the end of the first row, and probability $\frac{1}{3}$ of putting the 4 on the bottom row.

4. What are the probabilities of putting the 5 at the end of the first row or second row in

$$\begin{array}{ccc} \square & \square & \square \\ \square & \square & \end{array} ?$$

5. Continuing the preceding problem, what are the probabilities of each of the standard tableaux in the diagram of problem **4**?

I now describe the Murnaghan-Nakayama rule: A **skew hook** is a connected path on the diagram consisting of moves either to the west or south which can be removed so that the remaining boxes form a diagram. In this definition, the position of the boxes is unchanged, so that the top row is *not* a skew hook (unless the diagram consisted of only one row) , since removing it would leave a diagram with zero boxes on the top row and some non-zero number of boxes on the remaining rows. the number of boxes in the skew hook is k, then we call it a k-skew hook. Thus in the diagram

$$\begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \end{array}$$

there are exactly two 4-skew hooks:

$$\begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \bigcirc & \bigcirc \\ \square & \bigcirc & \bigcirc & \end{array} \quad \text{and} \quad \begin{array}{cccc} \square & \square & \square & \bigcirc \\ \square & \square & \bigcirc & \bigcirc \\ \square & \square & \bigcirc & \end{array} .$$

If a skew hook starts on the i-th row and ends on the j-th row then $i - j$ is called the **leg length** of the skew hook. Thus the leg lengths of the two 4-skew hooks above are 1 and 2.

Let λ be a Young diagram of S_n, and χ^λ its character. Let

$$\mu = (\mu_1, \mu_2 \ldots, \mu_r), \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$$

be the cycle structure of a conjugacy class in S_n We wish to compute the integer $\chi^\lambda(\mu)$. The Murnaghan-Nakayama rule says that

$$\chi^\lambda(\mu) = \sum (-1)^\ell \chi^{\lambda'}(\overline{\mu})$$

where the sum is over all μ_1-skew hooks, where ℓ is the leg length of the skew hook and where $\overline{\mu}$ is the partition of $n - \mu_1$ consisting of the remaining pieces of μ:

$$\overline{\mu} := (\mu_2, \ldots, \mu_n)$$

and where λ' is the Young diagram obtained from λ by removing the skew hook. The Murnaghan-Nakayama rule is a recursive rule for computing the desired character value. If there are no μ_1-skew hooks, then the above sum is interpreted as zero. For example, in the above diagram, the maximal number of boxes in a skew hook is six. So the corresponding character vanishes on any conjugacy class containing a cycle of length seven or greater. In using the rule iteratively, at the final stage of any branch in the iteration we end up either with zero or with the empty box and the empty partition which is interpreted as the number one. For example, if

$$\lambda = \begin{array}{lll} \square & \square & \square \\ \square & & \end{array}$$

then

$$\chi^\lambda((4)) = -1$$

since there is exactly one 4-skew hook and it has leg length 1,

$$\chi^\lambda((3,1)) = 0$$

since there are no 3-skew hooks.

$$\chi^\lambda((1,1,1,1)) = 3$$

since there are two 1-skew hooks both of leg length 0 - consisting of the box at the end of the first row and the box on the second row; removing the box at the end of the first row leaves the diagram (2,1), and this has two 1-skew hooks, and removing either one leaves a diagram with only one 1-skew hook, while removing the box on the second row of the original diagram leaves a unique choice at each further stage

6. Find the value of the character of the irreducible six dimensional representation of S_5 on all conjugacy classes using the Murnaghan-Nakayama rule.