

## Math 126 Exercise set 2.

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1. What is the character of the regular representation?
2. Let  $\square$  denote the set of vertices of the square, and label the vertices in cyclic order as  $\{1, 2, 3, 4\}$ . The group  $D_4$  acts on this 4 element set, and this gives rise to a representation of  $D_4$  on  $\mathcal{F}(\square)$  which is a four dimensional vector space. The one dimensional subspace of  $\mathcal{F}(\square)$  consisting of all constant functions is an invariant subspace, and the restriction of the representation on  $\mathcal{F}(\square)$  to this subspace is the trivial representation.
  1. Show that the one dimensional subspace consisting of all  $h$  which satisfy  $h(1) = h(3) = -h(2) = -h(4)$  is an invariant subspace, and compute the representation (and hence the character) of  $D_4$  acting on this subspace.
  2. Show that the two dimensional subspace of  $\mathcal{F}(\square)$  spanned by the functions  $f$  and  $g$  where  $1 = f(1) = f(2) = -f(3) = -f(4)$  and  $1 = g(2) = g(3) = -g(1) = -g(4)$  is an invariant subspace, and compute the character of the representation of  $D_4$  on this two dimensional subspace.
  3. Conclude that this two dimensional representation is irreducible.
  4. Conclude that  $D_4$  has two further irreducible representations, both one dimensional.

The group  $S_n$  acts on the set  $\{1, \dots, n\}$  and hence has a representation on the  $n$ -dimensional space  $\mathcal{F}(\{1, \dots, n\})$  known as the “permutation representation”. Let us (in this exercise set) denote this representation by  $p$ . We know that the constant functions form a one dimensional invariant subspace giving rise to the trivial representation, and that there is a complementary invariant subspace of dimension  $n - 1$  which is irreducible. Let us denote this representation by  $r_{n-1}$ . So

$$p = \text{trivial} \oplus r_{n-1}.$$

The map  $S_n \rightarrow \mathbb{C}$  given by

$$a \mapsto \det p(a)$$

us a one dimensional representation known as the sign representation and denoted by  $\text{sgn}$ . So

$$\text{sgn}(a) := \det p(a).$$

**3.** Show that  $\text{sgn}((12)) = -1$  so this representation  $\text{sgn}$  is not trivial.

**4.** Show that the  $\text{sgn}((12\dots k)) = (-1)^{k-1}$ . [Hint: use the factorization of the cycle  $(12\dots k)$  as

$$(12\dots k) = (1k)\cdots(13)(12)].$$

**5.** What is the character of  $r_{n-1}$  evaluated on the element  $(12)$ ?

**6.** Show that  $r_{n-1} \otimes \text{sgn}$  is irreducible, and is not equivalent to  $r_{n-1}$  if  $n > 3$ . What goes wrong with this argument when  $n = 3$ ?

**7.** Conclude that  $S_4$  has a two dimensional representation, and five irreducibles in all.

**8.** Construct the two dimensional irreducible representation of  $S_4$  by exhibiting a homomorphism from  $S_4$  onto  $S_3$ . [Hint. Exhibit a three element set on which  $S_4$  acts as all permutations.]