

Induced representations and homogeneous vector bundles.

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The space $\mathcal{F}(G, F)^{r, (H, \square)}$.

Let H be a subgroup of the group G , so that G acts transitively on $M:=G/H$. We identified the space $\mathcal{F}(M)$ as the subspace of $\mathcal{F}(G)$ consisting of those elements which are fixed under the right action of H . That is, we identified $\mathcal{F}(M)$ as the subspace of $\mathcal{F}(G)$ which transform as the trivial representation under H . If we are given any representation \square of H on a vector space F , we can consider the space $\mathcal{F}(G, F)$ of all functions on G with values in F . As usual, the group G acts on this space because of its action on itself:

$$(af)(b) := f(a^{-1}b).$$

We can consider the subspace $\mathcal{F}(G, F)^{r, (H, \square)}$ of $\mathcal{F}(G, F)$ consisting of those f which satisfy

$$f(bh) = \square(h)^{-1}f(b) \quad \square \quad h \in H \quad \text{and} \quad \square \quad b \in G.$$

Induced representations.

The space $\mathcal{F}(G, F)^{r, (H, \square)}$ is an invariant subspace of $\mathcal{F}(G, F)$ under the left action of G since left multiplication and right multiplication commute. So we get a representation of G on $\mathcal{F}(G, F)^{r, (H, \square)}$.

In other words, starting from a representation \square of the subgroup H we have obtained a representation of the group G , called the representation of G **induced** from the representation \square of H .

We want to understand this construction from a more geometric point of view.

Vector bundles.

Let M be a finite set. A *vector bundle* over M consists of a collection of vector spaces E_x , one vector space for each point x of M . We let $E = \bigcup_{x \in M} E_x$ (union, not direct sum) and sometimes talk of E as being the vector bundle. E is *not* a vector space. We define the map $\pi: E \rightarrow M$ by setting $\pi(v) = x$ if $v \in E_x$. That is, any $v \in E$ belongs to one of the vector spaces E_x , and we let $\pi(v)$ be this particular x . The space $E_x = \pi^{-1}(x)$ is sometimes called the *fiber* over x , Fig. 3.4(a).

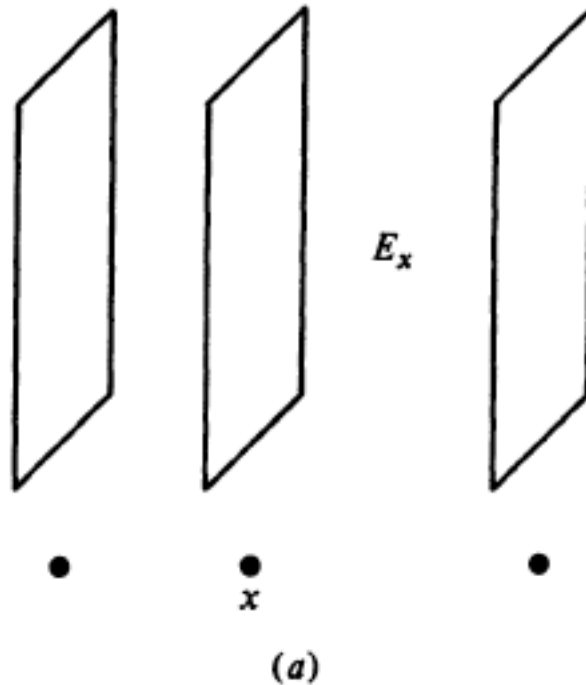
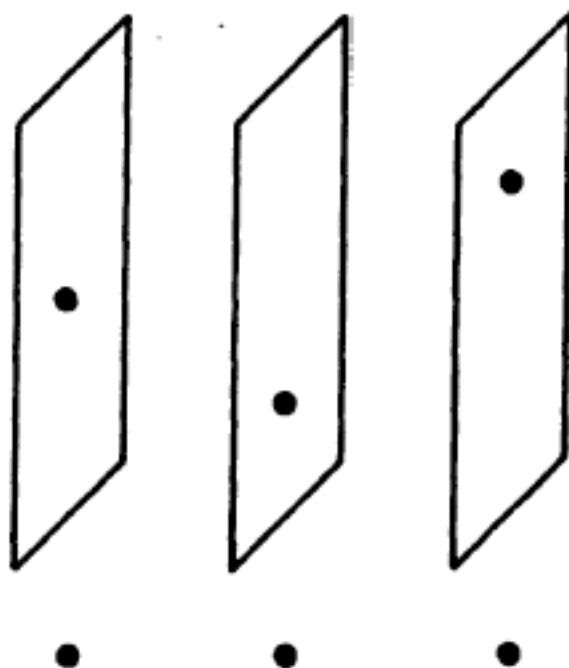


Fig. 3.4

Sections.

Let E be a vector bundle over M . A *section* of E is a function f , which assigns a vector $f(x) \in E_x$ to each $x \in M$. Thus, f is a map from M to E with the property that



The space of all sections is a vector space.

Let f_1 and f_2 be sections of the vector bundle E . Let x be a point of M . Since $f_1(x)$ and $f_2(x)$ take values in the same vector space E_x , it makes sense to add them. We thus define the sum of two sections by:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x).$$

Similarly, if c is any complex number and f is any section, we define cf by

$$(cf)(x) = cf(x).$$

In this way, *the space of all sections becomes a vector space*. We denote this vector space by $\Gamma(E)$.

Action of a group on a vector bundle.

Suppose that the group G acts on M and that E is a vector bundle over M . We say that G acts as a group of vector bundle morphisms on E , or that E is a *homogeneous vector bundle* under G , if

- (1) G acts on E ;
- (2) the map $\pi: E \rightarrow M$ is a G morphism, i.e. $a\pi(v) = \pi(av)$ for all $a \in G$ and $v \in E$.

Condition (2) is the same as saying that $a: E_x \rightarrow E_{ax}$. We require also that

- (3) the map $a: E_x \rightarrow E_{ax}$ is linear for each $a \in G$ and $x \in M$.

Thus, each a permutes the various vector spaces according to its action on M , and is a linear map from one vector space to another.

The representation of G on $\Gamma(E)$.

Suppose that the group G acts on M and also acts on the vector bundle $E \rightarrow M$. We define an action of G on $\Gamma(E)$ by setting

$$r(a)f(x) = a[f(a^{-1}x)]. \quad (2.1)$$

Notice that $f(a^{-1}x) \in E_{a^{-1}x}$ so that the right-hand side of (2.1) is an element of E_x , and thus $r(a)f$ is indeed again a section of E . It is clear that the map $f \mapsto r(a)f$ is a linear transformation, and it is easy to check that r is a representation of G on $\Gamma(E)$. We shall denote this representation by r_E , and the corresponding character by χ_E .

A \square basis of $\square(E)$.

In

order to evaluate χ_E we introduce a convenient basis of $\Gamma(E)$, generalizing the δ function basis we used for $F(M)$. For each vector space, E_x , we introduce a basis $\mathbf{v}_{x1}, \dots, \mathbf{v}_{xk_x}$, where $k_x = \dim E_x$. For any $\mathbf{v} \in E_x$ we define the section $f_{\mathbf{v}}$ by

$$f_{\mathbf{v}}(y) = \begin{cases} \mathbf{v} & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

It now follows, as for the case of $F(M)$, that

$$r_E(a)f_{\mathbf{v}} = f_{a\mathbf{v}}. \quad (2.2)$$

As x ranges over all of M and the \mathbf{v}_{xi} over a basis of each E_x , the sections $f_{\mathbf{v}_{xi}}$ form a basis of $\Gamma(E)$:

If we use this basis, it is clear from (2.2) that the only non-zero entries on the diagonal of the matrix representing $r_E(a)$ can come from those \mathbf{v}_{xi} for which $ax = x$.

The Frobenius character fixed point formula.

For each x with $ax = x$, the element a maps E_x into itself, and

$$af_{v_{xi}} = f_{av_{xi}}.$$

Now if (A_{xij}) is the matrix of the map $a: E_x \rightarrow E_x$ with respect to the basis v_{xi} , then

$$av_{xj} = \sum_i A_{xij} v_{xi}$$

and hence

$$af_{v_{xj}} = \sum_i A_{xij} f_{v_{xi}}.$$

Thus, for each fixed x the sum of the diagonal elements of the linear transformation $f \rightsquigarrow af$ coming from the v_{xi} is $\sum_i A_{xii}$, which is just the trace of the linear transformation $a: E_x \rightarrow E_x$. Summing over all the fixed points, x , of M , we obtain the celebrated

Frobenius fixed point character formula:

$$\chi_E(a) = \sum_{ax=x} [\text{tr}(a: E_x \rightarrow E_x)]. \quad (2.3)$$

This generalizes the fixed point formula that we obtained for the action of G on the space of functions on M .

Orbit decomposition.

Let $E \rightarrow M$ be a homogeneous vector bundle for the group G and let N be a subset of M . We can form $\pi^{-1}N = \bigcup_{x \in N} E_x$, which is now a vector bundle over N , which we denote by E_N . If N is mapped onto itself by all elements of G , i.e. is stable under G , then E_N is clearly a homogeneous vector bundle for G . We can identify $\Gamma(E_N)$ with a subspace of $\Gamma(E)$, namely the subspace consisting of all sections which vanish outside of N , i.e those f for which $f(y) = 0$ for $y \notin N$. If $M = \cup M_i$ is a disjoint union, then we have the direct sum decomposition

$$\Gamma(E) = \Gamma(E_{M_1}) \oplus \cdots \oplus \Gamma(E_{M_k}).$$

If the M_i are invariant, in particular if they are orbits, we obtain the Mackey decomposition formula:

$$r_E = r_{E_{M_1}} \oplus \cdots \oplus r_{E_{M_k}} \tag{3.1}_r$$

and the corresponding character decomposition

$$\chi_E = \chi_{E_{M_1}} + \cdots + \chi_{E_{M_k}}. \tag{3.1}_\chi$$

The transitive case.

So we might as well look at the case where G acts transitively on M . Let m be a point of M and $H = G_m$, the isotropy group of m . Each element of H maps E_m into E_m and acts linearly. In other words, we get a representation of H on E_m . Let us denote this representation by s . Let n be some other point of M . Since G acts transitively, we can write

$$n = am$$

for some $a \in G$, and if $n = bm$, then $b = ah$ for some $h \in H$. Any vector in E_n is the image of some vector in E_m under the action of a . In other words, we can write any vector in E_n as

$$\mathbf{w} = a\mathbf{u} \quad \mathbf{u} \in E_m.$$

If also

$$\mathbf{w} = b\mathbf{v}$$

then $b\mathbf{v} = a\mathbf{u}$ or $a^{-1}b\mathbf{v} = \mathbf{u}$ or

$$\mathbf{u} = s(h)\mathbf{v}.$$

This shows that the representation s determines the vector bundle E .

Constructing the vector bundle.

We now show

how to construct a homogeneous vector bundle from a representation of a subgroup.

So suppose we start with G and a subgroup H . We can reconstruct M as the coset space G/H : a point of $M = G/H$ is a coset aH . Suppose that s is a representation of the subgroup H on a vector space V . On the space $G \times V$ we introduce the equivalence relation

$$(gh, \mathbf{v}) \sim (g, s(h)\mathbf{v})$$

and let E denote the set of all equivalence classes. We denote the equivalence class of $(g, \mathbf{v}) \in G \times V$ by $[(g, \mathbf{v})]$. We write

$$E = G \times_H V.$$

The map sending $(g, \mathbf{v}) \rightsquigarrow gH \in M$ is clearly constant on equivalence classes, and hence defines a map from $E \rightarrow M$. Suppose that $x = gH$. Let E_x consist of all equivalence classes of elements (g, \mathbf{v}) as \mathbf{v} ranges over V . We have an identification of the set E_x with the vector space V , which depends on the choice of g :

$$\phi_g: V \rightarrow E_x, \quad \phi_g(\mathbf{v}) = [(g, \mathbf{v})].$$

Constructing the vector bundle, continued.

If we change our choice of g by replacing g by gh , we get

$$\phi_{gh}(\mathbf{v}) = [(gh, \mathbf{v})] = [(g, s(h)\mathbf{v})] = \phi_g(s(h)\mathbf{v})$$

so that

$$\phi_{gh} = \phi_g \circ s(h). \tag{3.2}$$

This equation shows that we may use the map ϕ_g to define a vector space structure on E_x which is independent of the particular choice of g . If $\mathbf{e}_i = \phi_g(\mathbf{v}_i)$, we define

$$\mathbf{e}_1 + \mathbf{e}_2 = \phi_g(\mathbf{v}_1 + \mathbf{v}_2).$$

In view of (3.2) and the fact that $s(h)$ acts linearly on V , the value of this sum does not depend on the choice of g , and similarly for the definition of multiplication of a vector by a scalar. We have thus made E into a vector bundle over M .

The action of G .

We define the action of the

group G on this vector bundle by multiplication on the left:

$$a[(g, \mathbf{v})] = [(ag, \mathbf{v})].$$

It is obvious that this definition is independent of the choice of representative. It defines a linear map from $E_x \rightarrow E_{ax}$ by the very definition of the linear structure on these fibers. Thus E is a homogeneous vector bundle for G , and we obtain a representation of G on $\Gamma(E)$. This representation was obtained from the subgroup H and the representation s of H . It is called the representation of G *induced* from the representation s , and we will denote it by $(s \uparrow G)$.

We will show that this is the same definition as we gave earlier in just a bit.

The induced character.

Let s be a representation

of the subgroup H of the group G , and let σ be its character. We shall denote the character of the induced representation, $s \uparrow G$, by $\sigma \uparrow G$. The formula for $\sigma \uparrow G$ is given by (2.3). It is sometimes convenient to rewrite (2.3) in a slightly different form. The sum in (2.3) is over fixed points. To say that $x = gH$ is fixed under $a \in G$ means that $g^{-1}ag \in H$. Of course the element g is only determined by x up to right multiplication by an arbitrary element of H . The action of a on E_x can be described as follows: if $\mathbf{u} \in E_x$ is given by $\mathbf{u} = \phi_g \mathbf{v}$ for $\mathbf{v} \in V$, then

$$\mathbf{u} = [(g, \mathbf{v})] \text{ so } a\mathbf{u} := [(ag, \mathbf{v})] = [(gg^{-1}ag, \mathbf{v})] = [(g, s(g^{-1}ag)\mathbf{v})].$$

Thus

$$\text{tr}[a: E_x \rightarrow E_x] = \text{tr } s_{g^{-1}ag} = \sigma(g^{-1}ag).$$

The expression on the right makes sense because $g^{-1}ag \in H$, and does not depend on which g we pick with $x = gH$. For purposes of counting, it is sometimes easier to sum over *all* g with $g^{-1}ag \in H$, instead of summing over the fixed points, but then we will have counted each fixed point $\#H$ times. We divide by $\#H$ to compensate for the overcounting, and (2.3) becomes

$$\sigma \uparrow G(a) = (1/\#H) \sum_{\substack{g \in G \\ g^{-1}ag \in H}} \sigma(g^{-1}ag). \quad (3.3)$$

The Frobenius reciprocity formula.

Let χ be a character of the group G , and let $\chi|_H$ denote the restriction of χ to H . Let us compute $(\sigma \uparrow G, \chi)_G$. We have

$$\begin{aligned}(\sigma \uparrow G, \chi)_G &= \frac{1}{\#G} \sum (\sigma \uparrow G)(a) \overline{\chi(a)} \\ &= \frac{1}{\#G} \frac{1}{\#H} \sum_{\substack{a \in G \\ h = b^{-1}ab \in H}} \sigma(b^{-1}ab) \overline{\chi(a)} \text{ by (3.3)} \\ &= \frac{1}{\#G} \frac{1}{\#H} \sum_{\substack{h \in H \\ b \in G}} \sigma(h) \overline{\chi(bhb^{-1})} \\ &= \frac{1}{\#H} \sum_{h \in H} \sigma(h) \overline{\chi(h)}\end{aligned}$$

since $\chi(bhb^{-1}) = \chi(h)$

$$= (\sigma, \chi|_H)_H.$$

Thus we have proved the *Frobenius reciprocity formula*:

$$(\sigma \uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \tag{3.4}$$

More on Frobenius reciprocity.

Thus we have proved the *Frobenius reciprocity formula*:

$$(\sigma \uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \quad (3.4)$$

If χ is the character of a representation of G on a vector space W , the left-hand side of (3.4) is just $\dim \text{Hom}_G(W, \Gamma(E))$. If σ is the character of a representation of H on a vector space F (so the fiber of E over H is F), the right-hand side of (3.4) is just $\dim \text{Hom}_H(W, F)$. Thus we can rewrite (3.4) as

$$\dim \text{Hom}_G(W, \Gamma(E)) = \dim \text{Hom}_H(W, F). \quad (3.5)$$

In fact, we can say more: that there is a natural identification of the two vector spaces $\text{Hom}_G(W, \Gamma(E))$ and $\text{Hom}_H(W, F)$.

The two constructions of induced representations.

Let $x \in M = G/H$. Recall that for each $b \in G$ such that $x = bH$, we have identified E_x with the set of all (b, v) , where $v \in F$. If $f \in \Gamma(E)$, then $f(x) \in E_x$, so we can write

$$f(x) = [(b, \hat{f}(b))],$$

where $\hat{f}(b) \in F$. If $c \in H$, then $f(x) = f(bH) = f(bcH)$, and

$$f(bcH) = [(bc, \hat{f}(bc))].$$

Thus

$$\hat{f}(bc) = s(c)^{-1} \hat{f}(b) \quad \text{for } c \in H. \quad (3.6)$$

Conversely, any function $\hat{f}: G \rightarrow F$ satisfying (3.6) defines a section of $\Gamma(E)$. Thus we may identify $\Gamma(E)$ with the space of all functions from G to F satisfying (3.6). Let us denote this space by $\hat{\Gamma}$. We now compare the action of G on both spaces. On $\Gamma(E)$, the representation r_E is given by

$$(r_E(a)f)(x) = af(a^{-1}x)$$

while on the space of functions the representation \hat{r} is given by

$$\hat{r}(a)\hat{f}(g) = \hat{f}(a^{-1}g).$$

If $x = bH$, then

$$f(x) = [(b, \hat{f}(b))]$$

and

$$\begin{aligned} (r_E(a)f)(x) &= a[(a^{-1}b, \hat{f}(a^{-1}b))] \\ &= [(b, \hat{f}(a^{-1}b))] \end{aligned}$$

so the function corresponding to $r_E(a)f$ is $\hat{r}(a)\hat{f}$ as required.