

MATH 126: MIDTERM

This midterm is due at the start of class on Wednesday November 8. You may not work with others. You may use class notes, M. Artin's book "Algebra" and Serre's book "Linear representations of finite groups", but no other resource. You may not consult any other literature. With your solutions please include a signed statement that you have complied with these requirements. GOOD LUCK!

All vector spaces are assumed to be finite dimensional over an algebraically closed field of characteristic 0. All groups are assumed to be finite.

1) a) If G is a simple group show that G has an irreducible faithful representation.

b) If a group G has an irreducible faithful representation show that the centre of G is cyclic.

2) a) If G is a finite group we let $[G, G]$ denote the normal subgroup generated by all products $ghg^{-1}h^{-1}$ with $g, h \in G$. Show the number of 1-dimensional representations of G equals the order of $G/[G, G]$.

b) If G is any group of order 32 with 17 conjugacy classes find the dimensions of the irreducible representations of G .

3) Suppose that a finite group G acts on a finite set Ω . Let $V_\Omega \cong k^\Omega$ denote the corresponding permutation representation. Recall that we showed that $\chi_{V_\Omega}(g)$ was equal to the number of fixed points of g on Ω . Let χ_0 denote the character of the trivial representation of G .

a) Show that

$$(\chi_{V_\Omega}^\vee, \chi_0) = (\#G)^{-1} \sum_{x \in \Omega} \#\text{Stab}_G(x),$$

which in turn equals the number, $h(\Omega)$, of orbits of G in Ω . We write $V_\Omega = W_\Omega \oplus k^{h(\Omega)}$.

b) Show that $V_\Omega^\vee \cong V_\Omega$.

c) Let G act on $\Omega \times \Omega$ by $g(x, y) = (gx, gy)$. Show that $V_{\Omega \times \Omega} = V_\Omega \otimes V_\Omega$.

d) By calculating $(\chi_{V_\Omega}, \chi_{V_\Omega}) = (\chi_{V_\Omega} \chi_{V_\Omega}^\vee, \chi_0)$ in two ways show that

$$(\chi_{W_\Omega}, \chi_{W_\Omega}) = h(\Omega \times \Omega) - h(\Omega)^2.$$

e) Suppose that G acts *doubly transitively* on Ω , i.e. for all $x, y, x', y' \in \Omega$ with $x \neq y$ and $x' \neq y'$ there exists $g \in G$ with $gx = x'$ and $gy = y'$. Show that V_Ω is the sum of two irreducible representations, one of which is trivial.

4) Find the character table of S_5 . Explain your reasoning.

5) Let G be a finite group, let $[g_1], \dots, [g_h]$ be its conjugacy classes and let χ_1, \dots, χ_h denote the characters of its irreducible representations. Also let $\sigma : G \rightarrow G$ be an automorphism, i.e. a bijective homomorphism from G to itself. For all i show that $\sigma[g_i]$ is a conjugacy class in G and that $\chi_i \circ \sigma$ is the character of an irreducible representation of G .

Show that the number of i such that $\chi_i = \chi_i \circ \sigma$ equals the number of j such that $\sigma([g_j]) = [g_j]$.