MATH 126 PROBLEM SET 6: $GL_2(\mathbb{F}_p)$ AND INDUCTION

This problem set is due Wednesday November 22. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0.

1) Let p be an odd prime. If φ is a character of $\mathbb{F}_{p^2}^\times$ with $\varphi \neq \varphi^p$, decompose $\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{GL_2(\mathbb{F}_p)} \phi$ into irreducibles.

2) Let p be a prime greater than 3. Let $SL_2(\mathbb{F}_p)$ denote the kernel of $\det : GL_2(\mathbb{F}_p) \to \mathbb{F}_p^\times$. If ρ is an irreducible representation of $GL_2(\mathbb{F}_p)$ show that either $\ker \rho \supset SL_2(\mathbb{F}_p)$ or $\ker \rho \subset \mathbb{F}_p^\times$. Deduce that any normal subgroup of $GL_2(\mathbb{F}_p)$ either contains $SL_2(\mathbb{F}_p)$ or is contained in $\mathbb{F}_p^\times$. Conclude that $SL_2(\mathbb{F}_p)/\{\pm 1\}$ is a simple group.

3) Let p be an odd prime, let $N \subset GL_2(\mathbb{F}_p)$ be the subgroup consisting of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and let $\psi : N \to \mathbb{C}^\times$ be the representation

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \longmapsto e^{2\pi i b/p}.$$

Decompose $\mathrm{Ind}_N^{GL_2(\mathbb{F}_p)} \psi$ into irreducibles. [Optional: What unusual phenomenon do you notice?]

4) Suppose that A is an abelian subgroup of G and that (ρ, V) is an irreducible representation of G. Show that $\dim V \leq \#G/\#A$. [HINT: Consider ρ restricted to A.]

5) Show that A_4 is the semi-direct product of a group of order 4 by $\langle (123) \rangle$. Hence find the irreducible representations of A_4.

6) Let p denote a prime number and let B denote the subgroup of $GL_2(\mathbb{F}_p)$ consisting of upper triangular matrices. Find all irreducible representations of B. [HINT: B is the semidirect product of N (see question 3) and the subgroup T of diagonal matrices.]

7) Let $\Omega(G)$ denote the set of cyclic subgroups of a group G. Show that

$$\sum_{H \in \Omega(S_3)} \mathrm{Ind}_H^{S_3} R(H) = R(S_3),$$

but that

$$\sum_{H \in \Omega(A_4)} \mathrm{Ind}_H^{A_4} R(H) \neq R(A_4).$$

8) Let Ω denote the set of cyclic subgroups of G. Let V denote the $\mathbb{Q}$-vector space with basis the set of pairs (H, ϕ), where $H \in \Omega$ and $\phi \in \widehat{H}$. Let W denote the subspace of V generated by

- $(H, \phi) - (gHg^{-1}, \phi^g)$ for all $H \in \Omega$, $\phi \in \widehat{H}$ and $g \in G$, and
- $(H', \phi) - \sum_i (H, \phi_i)$ for all $H \in \Omega$, $H' \subset H$ and $\phi \in \widehat{H'}$ with $\mathrm{Ind}_{H'}^{H} \phi \cong \oplus_i \phi_i$.

Show that the map $V \to R(G)_{\mathbb{Q}}$ which sends $(H, \phi) \mapsto \mathrm{Ind}_H^G \phi$ induces an isomorphism $V/W \xrightarrow{\sim} R(G)_{\mathbb{Q}}$. [HINT: Show that it suffices to prove the same result with $\mathbb{C}$ replacing $\mathbb{Q}$. Then rephrase the problem in terms of spaces of class functions and use duality.]