MATH 126 PROBLEM SET 3: DEFINITIONS AND CONSTRUCTIONS

This problem set is due Wednesday, October 18th. In questions 1) thru 4) we suppose that we are working over a field of characteristic 0.

Recall that $S_3 \cong \langle a, b : a^3 = b^2 = abab = 1 \rangle$, where a corresponds to (123) and b corresponds to (12). For $i = 0, 1, ..., 5$ let (ρ_i, V_i) be the follwoing represnetations of S_3:

- $\rho_0(a) = 1$, $\rho_0(b) = 1$;
- $\rho_1(a) = 1$, $\rho_1(b) = -1$;
- $\rho_2(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho_2(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
- $\rho_3(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- $\rho_4(a) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $\rho_4(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- $\rho_5(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\rho_1(b) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

1) Show that $V_5 \cong V_0 \oplus V_4$. [Hint: Consider the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ of V_5.]

2) Show that $S^2V_4 \cong V_5$. [Hint: Consider the basis $e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2$ of S^2V_4.]

3) Show that the induction from $\{1, (12)\}$ to S_3 of the trivial representation is isomorphic to V_5.

4) Show that the induction from $\{1, (12)\}$ to S_3 of the sgn representation is isomorphic to $V_1 \oplus V_4$. [Hint: Consider the map

$$f \longmapsto \begin{pmatrix} f(1) + f(123) + f(132) \\ f(1) - f(132) \\ -f(1) + f(123) \end{pmatrix}.]$$

5) If $\rho_b : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ is the representation defined by $\rho_b(a) = e^{2\pi iab/n}$, show that

$$\rho_{b_1} \otimes \rho_{b_2} \cong \rho_{b_1+b_2}.$$

6) Show that $\mathrm{Hom}_G(U, V \oplus W) \cong \mathrm{Hom}_G(U, V) \oplus \mathrm{Hom}_G(U, W)$.

7) If G acts transitively on a set Ω and if $x \in \Omega$, show that the induction from $\mathrm{Stab}_G(x)$ to G of the trivial representation is isomorphic to the permutation representation of G on k^Ω.

8) Let Q denote the group $\langle i, j : i^2 j^2 = i^4 = ijij^3 = 1\rangle$. Show that Q has order 8 and write down an irreducible two dimensional representation of Q over $\mathbb{C}$. [Hint: Look for a representation in which i^2 maps to minus the identity matrix.]