MATH 126 PROBLEM SET 8: BRAUER'S THEOREM (OPTIONAL)

This problem set is optional, i.e. not for credit. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0.

1) Show that a subgroup of an elementary group is elementary. Also show that every elementary group is nilpotent.

2) Suppose that $m|n$ are integers with $m > 1$ and let ζ be a primitive m^{th} root of 1. Let $\psi : (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}^{\times}$ be the one dimensional representation sending $1 \mapsto \zeta$. By computing (χ_0, ψ) two ways show that

$$\sum_{a \in \mathbb{Z}/n\mathbb{Z}} \zeta^a = 0.$$

3) Show that the 3-dimensional irreducible representations of S_4 are induced from a 1-dimensional representation of a Sylow 2-subgroup of S_4. Write all irreducible representations of S_4 as $\mathbb{Z}$-linear combinations of representations induced from 1-dimensional representations of elementary subgroups.

4) Write all irreducible representations of A_5 as $\mathbb{Z}$-linear combinations of representations induced from 1-dimensional representations of elementary subgroups.

5) If G is a finite group let $R_0(G) \subset R(G)$ denote the $\mathbb{Z}$-span of the elements of the form $\mathrm{Ind}_H^G(\alpha - 1)$, where $H < G$ and where α is a 1-dimensional representation of G. Show that if $\theta \in R_0(G)$ then $\theta(1) = 0$. In this question we will prove that $R(G) = \mathbb{Z}\chi_0 \oplus R_0(G)$, i.e. that $\theta \in R_0(G)$ if and only if $\theta(1) = 0$.

First consider the case that G is an elementary group, and argue by induction on $\#G$. The case $\#G = 1$ is trivial. Let Ω denote the set of maximal normal (proper) subgroups of G. Show that if $H \in \Omega$ then G/H is cyclic of prime order. Also show that $R(G)$ is spanned by the 1-dimensional representations of G together with $\mathrm{Ind}_H^G R(H)$ for $H \in \Omega$. Use the inductive hypothesis to deduce that $R(G)$ is also spanned by the 1-dimensional representations of G together with $\mathrm{Ind}_H^G R_0(H)$ for $H \in \Omega$. conclude that $R(G) = \mathbb{Z}\chi_0 \oplus R_0(G)$.

Now let G be any finite group. Show that we can find $\psi_H \in R(H)$ for $H \in \Omega^{ex}$ such that

$$\chi_0 = \sum_{H \in \Omega^{ex}} \mathrm{Ind}_H^G \psi_H$$

and hence that for any $\theta \in R(G)$ we have

$$\theta = \sum_{H \in \Omega^{ex}} \mathrm{Ind}_H^G (\psi_H \theta).$$

Deduce that if $\theta(1) = 0$ then $\theta \in R_0(G)$.

6) Let Ω be a set of subgroups of a finite group G such that

$$R(G) = \sum_{H \in \Omega} \operatorname{Ind}_H^G R(H).$$

In this question we will prove that every elementary subgroup of G is conjugate to a subgroup of some element of Ω.

To this end let E be a maximal p-elementary subgroup of G. Thus $E = \langle g \rangle \times P$, where g has order prime to p and where P is a p-group. Show that P is a Sylow p-subgroup of $Z_G(g)$. Let $H \in \Omega$ and ψ be a class function on H with integral values. Also let $[g]_G \cap H$ be the disjoint union of H-conjugacy classes $[h_1]_H, \ldots, [h_r]_H$. Show that

$$(\operatorname{Ind}_H^G \psi)(g) = \sum_{i=1}^{r} \#(Z_G(h_i)/Z_H(h_i))\psi(h_i).$$

Also show that if for some i, $p \not| \#(Z_G(h_i)/Z_H(h_i))$ then H contains a G-conjugate of E (for instance the product of $\langle h_i \rangle$ and a Sylow p-subgroup of $Z_H(h_i)$). Deduce that

$$(\operatorname{Ind}_H^G \psi)(g)/p$$

must be integral.

Conclude that

$$\chi_0 \notin \sum_{H \in \Omega} \operatorname{Ind}_H^G R(H).$$

7) Show that

$$R(G) = \sum_{H \in \Omega^{cy}} \operatorname{Ind}_H^G R(H)$$

if and only if every Sylow subgroup of G is cyclic.

8) Decompose into irreducibles S^3 and $\wedge^3$ of each of the irreducible 3-dimensional representations of S_4.