MATH 126 PROBLEM SET 1

Due on Thursday Sep.20-th

1. Prove that for any group G the center Z(G) is a normal subgroup.

2. Let D_n be the group generated by two generators a and b and relations $a^n = b^2 = baba = e$ where $e \in D_n$ is the identity.

a)Find the order of the group D_n,

b) describe the center Z(D_n)

c)find all subgroups of D_n and decide which ones are normal.

A solution. For any divisor m of n we have subgroups $L_m := \{a^{km}\}, 1 \leq k \leq n/m$ and $R_m := L_m \cup bL_m$. The subgroups L_m are all normal, the subgroup R_m is normal only when $m = 2, [R_1 = D_n]$

3.a) Find all conjugacy classes in the group

$$P(\mathbb{F}_p) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad , a \in \mathbb{F}_p^*, b \in \mathbb{F}_p$$

where p is a prime number and $\mathbb{F}_p$ is the field of residius mod p.

b) Is the group $P(\mathbb{F}_p)$ solvable?

4.a) What are the orders of the various conjugacy classes in S_4? List all normal subgroups of S_4. Is S_4 solvable?

b) construct a surjective homomorphism $\phi : S_4 \to S_3$.

A solution. Let Y be set of all ways to divide $1, 2, 3, 4$ into two pairs. It is clear that the group S_4 acts naturally on the set Y. On the other hand you can check that has three elements $(12)(34), (13)(24), (14)(23)$

5. Let G be a finite abelian group. Prove that any irreducible representation of G is 1-dimensional.

6.Find all 1-dimensional representations of the group $S_n, n > 1$

A solution. Let $s_i \in S_n 1 \leq i \leq n-1$ be the permutation $i \leftrightarrows i+1$. Then $s_i^2 = Id$ and $s_{i+1} = g_i s_i g_i^{-1}$ for $g_i := s_i s_{i+1}$. Therefore for any 1-dimensional representation ρ we have $\rho(s_i) = c$ where either $c = 1$ [the trivial representation] or $c = -1$. One can show that there exist one-dimensional representation $\epsilon : S_n \to \mathbb{C}^*$ such that $\epsilon(\sigma) = (-1)^{l(\sigma)}, \sigma \in S_n$ where $l(\sigma)$ is the number of "disorders" in σ That is

$$l(\sigma) = \#\{(i,j) | 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$$

7. Let $X := \{1, ..., n\}, \rho : S_n \to Aut\mathbb{C}(X)$ be the permutation representation. Describe all S_n-invariant subspaces $W \subset \mathbb{C}(X)$.

8. a) Suppose that p is a prime number and that G is a finite group of p-power order which acts on a finite set Ω. Let $\Sigma \subset \Omega$ denote the subset of points fixed by G. Show that

$$\#\Sigma \equiv \#\Omega \mod p.$$

b) If G is a finite group of prime power order show that $Z(G)$ is non-trivial. Deduce that G is nilpotent.

9) Let V_1, V_2 be finite-dimensional $\mathbb{C}$-vector spaces. Serre [Section 1.5] characterizes the *tensor product* $V_1 \otimes V_2$ as a pair $(W, \phi : V_1 \times V_2 \to W)$ where W is a $\mathbb{C}$-vector space and ϕ is a map satisfying conditions (i) and (ii). Then Serre says "it is easily shown that such a space exists, and is unique"

Please a) show that the space $V_1 \otimes V_2$ exists

b) explain in what sense is this pace unique and prove the uniqueness of the space $V_1 \otimes V_2$

10)Construct a *canonical* isomorphism A of the linear space $(V \otimes W)^\vee$ with the linear space$\mathcal{B}(V, W)$ of bilinear forms on $V \times W$ with values in L.

[In our case *canonical* :=the construction of A does not require a choice of a basis in V.]