MATH 124 HOMEWORK #7

INNA ZAKHAREVICH

(1) Suppose that we had an equilateral triangle with its vertices on integer coordinates. We can assume without loss of generality that these coordinates are (a, b) and (c, d), with $a, b, d > 0$, $a > c$ and $b < d$. (This means that the triangle has one vertex on the origin, both other vertices in the upper half-plane, and at least one vertex in the positive quadrant.)

From the fact that the triangle is equilateral we know that

$$a^2 + b^2 = c^2 + d^2 = (a - c)^2 + (b - d)^2.$$

Subtracting $a^2 + b^2 + c^2 + d^2$ from each part of the equation and multiplying by -1 we get

$$a^2 + b^2 = c^2 + d^2 = 2(ac + bd).$$

Notice that if each of a, b, c, d is even we must have another triangle with vertices $(a/2, b/2)$ and $(c/2, d/2)$, so we can assume that at least one is odd. But then all of them must be odd, so each is either 1 or 3 mod 4. But then consider $ac + bd$. It needs to be odd, since $a^2 + b^2 \equiv c^2 + d^2 \equiv 2 \pmod 4$. However, it is the sum of two odd numbers, which means that it will be even. Contradiction. Thus these equations can't be satisfied in integers, and therefore there are no equilateral triangles with integer sides.

(2) We want to find all solutions for $x^2 + y^2 = 2z^2$. First, notice that either both x, y are even or both are odd. Suppose that x, y are both even. Then we can write $x = 2x'$ and $y = 2y'$. Plugging this in, we get that $2x'^2 + 2y'^2 = z^2$, which means that z is also even and so (x, y, z) was not a primitive solution. So we assume that x, y are both odd. If we write $x + y = 2n$ and $x - y = 2m$ we can write

$$(2z)^2 = 2(x^2 + y^2) = (x + y)^2 + (x - y)^2 = (2p)^2 + (2q)^2.$$

But this just means that (p, q, z) is a Pythagorean triple. Thus the solutiosn to this equation are

$$t\,(m + n, m - n, 2r)\,,$$

where (m, n, r) is a Pythagorean triple. Since all such triples are of the form $t(a^2 - b^2, ab, a^2 + b^2)$ we have that the solutions to the given equation are

$$t(a^2 - b^2 + ab, a^2 - b^2 - ab, 2(a^2 + b^2)).$$

(3) Consider $x^2 + 2y^2 = 3z^2$. Notice that if either x or y is divisible by 3 then so is the other one, and then the left-hand side is divisible by 9, so z is then also divisible by 3. So we can assume that neither x nor y is divisible by 3. Also, notice that x and z must be the same parity, and if both are even then y is also even. Thus we can assume that both x and z are odd. This implies that $2y^2 \equiv 2 \pmod 4$, which means that y is also odd.

Write $x + y = 2p$ and $x - y = 2q$; one of these must be 0 mod 3, and the other must not be. Without loss of generality, write $p = 3r$. Then we have

$$(3r + q)^2 + 2(3r - q)^2 = 3z^2 \Longrightarrow 27r^2 - 6rq + 3q^2 = 3z^2,$$

which means that

$$(q - r)^2 + 8(r)^2 = z^2.$$

Thus if (a, b, c) is a solution to $x^2 + 8y^2 = z^2$, then $(4b + a, 2b - a, c)$ is a solution to $x^2 + 2y^2 = 3z^2$. So now we need to characterize solutions to $x^2 + 8y^2 = z^2$. We can classify these into two cases: when x, z are even, and when they are odd. If both x and z are even, then a solution to this is a solution to $x^2 + 2y^2 = z^2$. Note that here, for x, y, z to be a primitive solution, we must have all three being odd. Then we get $2y^2 = (x - z)(x + z)$. Notice that

$$(x + z, x - z) = (2x, x - z) = 2^\epsilon(x, x - z) = 2^\epsilon(x, z)$$

where $\epsilon = 1$ if $2|x-z$ and 0 otherwise. We must have $(x, z) = 1$, so we get that $x-z = (2r)^2$ and $x+z = 2s^2$. Then we have that the solutions to the equation $x^2+8y^2 = z^2$, when both x and z are even, are $x = 2t(s^2 + 2r^2), y = trs, z = 2t(s^2 - 2r^2)$.

Now suppose that x and z are odd. We have $8y^2 = (x - z)(x + z)$. Both of these are even, and one must be divisible by 4. We can write without loss of generality that $x + z = 4a$ and $x - z = 2b$. Then we have $y^2 = ab$. By the above computation, we get that $(a, b) = 1$, so $a = r^2$ and $b = s^2$. Thus we have the same solutions as above, without the factor of 2. Thus we see that the solutions to the equation $x^2 + 8y^2 = z^2$ are of the form

$$t(r^2 - 2s^2, rs, r^2 + 2s^2).$$

Then the solutions to the given equation are

$$t(4rs + r^2 + 2s^2, 4rs - r^2 + 2s^2, r^2 + 2s^2).$$

(4) Consider each equation modulo 5. Then we need $\pm 2x^2 = z^2 \pmod 5$, which is soluble only if $x \equiv z \equiv 0 \pmod 5$. But then $25|5y^2$, which means that $5|y$, and $(x, y, z) \geq 5 > 1$. Thus there are no primitive solutions, which in turn means that the only solution in integers is the trivial solution.