MATH 124 HOMEWORK #5

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(1) (a) Consider first the number $\beta = \langle 4, 8, 4, 8, 4, \ldots \rangle$. We know that

$$\beta = 4 + \frac{1}{8 + \frac{1}{\alpha}}$$

so, solving for β and taking the positive root we get that

$$\beta = \frac{4 + 3\sqrt{2}}{2}.$$

Then $\alpha = 4 + 1/\beta$, so $\alpha = 3\sqrt{2}$.

(b) Note that α becomes periodic at a_2, so we have $r = 1$. Thus r is odd, so we have no solution to the first equation.
We have the equations $x^2 - (\alpha y)^2 = \pm 1$. This means that the solution will be given by $x = p_r, y = q_r$, where p_r/q_r is the r-th convergent. We have

$$(p_0, q_0) = (4, 1) \qquad (p_1, q_1) = (17, 4)$$

so the solution is $x = 17$, $y = 4$.

(2) (a) Notice that

$$x^2 - dy^2 = 1 \iff (x - \sqrt{d}y)(x + \sqrt{d}y) = 1,$$

which means that $x + \sqrt{d}y$ is a unit in $\mathbb{Z}[\sqrt{d}]$ whose inverse is of the form $x - \sqrt{d}$. We know that $a + b\sqrt{d}$ is a unit whose inverse is of the form $a - \sqrt{d}$. But then the inverse of $(a + b\sqrt{d})^n$ is $(a - b\sqrt{d})^n$, and these numbers are related by the proper relation (since $\sqrt{d}$ appears exactly when b is taken to an odd power, which means it will be negative). Thus the numbers x, y defined by $(a + b\sqrt{d})^n$ are also all units of the proper form, and thus are all solutions to the given equations.

(b) (i) Dividing the given equation by $(a + b\sqrt{d})^m$ (which is clearly positive) we get that

$$1 < \frac{s + t\sqrt{d}}{(a + b\sqrt{d})^m} < a + b\sqrt{d}.$$

We now simply need to show that there are integers u, v such that

$$\frac{s + t\sqrt{d}}{a + b\sqrt{d}} = u + v\sqrt{d}$$

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We know that $(a + b\sqrt{d})^m (a - b\sqrt{d})^m = 1$. Thus

$$\frac{s + t\sqrt{d}}{(a + b\sqrt{d})^m} = \frac{s + t\sqrt{d}}{(a + b\sqrt{d})^m} \frac{(a - b\sqrt{d})^m}{(a - b\sqrt{d})^m} = (s + t\sqrt{d})(a - b\sqrt{d})^m$$

which clearly gives integral u, v.

(ii) To show this part all we need to show is that $(u+v\sqrt{d})^{-1} = u-v\sqrt{d}$. But notice that

$$(u+v\sqrt{d})^{-1} = (s+t\sqrt{d})^{-1}(a-b\sqrt{d})^{-m} = (s-t\sqrt{d})(a+b\sqrt{d})^m$$

which is clearly the "conjugate" of $u+v\sqrt{d}$, so (u,v) is a solution to the given equation.

(iii) We know that $u+v\sqrt{d} > 1 > 0$, so it is positive, so $(u+v\sqrt{d})^{-1} > 0$.

(iv) But this means that we have found a solution that is smaller than (a,b) (by part (i)) and positive, which means that we have found a positive solution smaller than the minimal one. Contradiction. Thus there cannot be such s,t, so all solutions are of the form $(a+b\sqrt{d})^m$.

(3) (a) We want to show that for infinitely many n there is an m such that

$$n(n+1) = 2m^2.$$

Multitiplying both sides by 4 we get that this equivalent to there being infinitely many solutions to

$$(2n+1)^2 - 2m^2 = 1.$$

(Note that since for each n there are at most two solutions for m, infinitely many solutions implies infinitely many positive values for n.)
So we want to show that there are infinitely many solutions to the equation $x^2 - 2y^2 = -1$. (Note that any positive solution for x will give a positive n, since for this to work x must be odd.) Notice that if we have one solution (a,b) then all numbers x,y such that $x+\sqrt{d}y = (a+b\sqrt{d})^{2k+1}$ will produce solutions to the equation. Thus if we find one solution we are done. Consider the ordered pair $(1,1)$; this is a solution to the above equation. So we are done.

(b) We have shown in the previous part that thre are infinitely many square and triangular numbers. Thus we just need to find a formula for all triangular numbers. We know that any solution to the above equation will be of the form $(1+\sqrt{(2)})^{2k+1}$; we want to find the coefficient of $\sqrt{2}$ in the expansion. But

$$(1+\sqrt{2})^{2k+1} = \sum_{i=0}^{2k+1}\binom{2k+1}{i}\sqrt{2}^i = \sum_{i=0}^{k}\binom{2k+1}{2i}2^i + \sqrt{2}\sum_{i=0}^{k}\binom{2k+1}{2i+1}2^i.$$

Thus we know that the square triangular numbers are of the form

$$\sum_{i=0}^{k}\binom{2k+1}{2i+1}2^i.$$

(4) (a) Notice that

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 = (ad+bc)^2 + (ac-bd)^2.$$

So for most a,b,c,d we will have two different decompositions into a sum of two squares.

(b) Suppose that we have two different decompositions

$$p = P^2 + Q^2 = R^2 + S^2.$$

First, notice that $(P,Q) = (R,S) = 1$.We can assume WLOG that $P \equiv R \equiv 1 \pmod 2$ and $S \equiv Q \equiv 0 \pmod 2$

$$P^2 - R^2 = S^2 - Q^2,$$

$$(P-R)(P+R) = (S-Q)(S+Q).$$

Note that each of these factors must be even, so we can write

$$\frac{P-R}{2}\frac{P+R}{2} = \frac{S-Q}{2}\frac{S+Q}{2},$$

with each of the fractions an integer.
Let

$$a = \left(\frac{P-R}{2}, \frac{S-Q}{2}\right) \qquad b = \left(\frac{P+R}{2}, \frac{S+Q}{2}\right)$$
$$c = \left(\frac{P-R}{2}, \frac{S+Q}{2}\right) \qquad d = \left(\frac{P+R}{2}, \frac{S-Q}{2}\right).$$

We will show that $P^2 - Q^2 = 4abcd$. In particular, it is enough to show that $\frac{1}{4}(P^2 - Q^2) = abcd$. First, notice that since $\frac{S-Q}{2} | \frac{P^2-Q^2}{4}$ we must have

$$\frac{S-Q}{2} \left| \left(\frac{P-Q}{2}, \frac{S-Q}{2}\right) \left(\frac{P+Q}{2}, \frac{S-Q}{2}\right) = ad.\right.$$

In particular, we have that $ad = m\frac{S-Q}{2}$, for some integer m. Where can the m come from? It must come from primes that divide ac and $\frac{S-Q}{2}$, and those must divide ac more times than it divides $\frac{S-Q}{2}$, which means that those primes must divide *both* $\frac{P+R}{2}$ and $\frac{P-R}{2}$. Let q be such a prime, and write

$$\frac{P-R}{2} = q^{\alpha_1} m_1 \qquad \frac{P+R}{2} q^{\alpha_2} m_2 \qquad \frac{S-Q}{2} = q^{\alpha_3} m_3$$

where $q \not| m_i$. We know that $\alpha_3 \leq \alpha_1 + \alpha_2$, and that $q^{\alpha_1+\alpha_2-\alpha_3} | \frac{S+Q}{2}$ (since $\frac{P+R}{2}\frac{P-R}{2} = \frac{S+Q}{2}\frac{S-Q}{2}$). Notice that if a prime divides both $\frac{P+R}{2}$ and $\frac{P-R}{2}$ then it divides both P and R, and similarly for S and Q. However, this means that we must have $\alpha_1 + \alpha_2 = \alpha_3$, since if a prime divides P, Q, R, S then it must divide p, which is a contradiction. Thus we see that $q \not| m$. Since this holds for all primes, we see that $m = 1$, and $ad = \frac{S-Q}{2}$.
Analogously, we see that

$$bc = \frac{S+Q}{2} \qquad ac = \frac{P-R}{2} \qquad bd = \frac{P+R}{2}.$$

Then we know that

$$P^2 - R^2 = S^2 - Q^2 = 4abcd.$$

However, we also know that

$$
\begin{aligned}
ac + bd &= \frac{P-R}{2} + \frac{P+R}{2} = P \\
ad - bc &= \frac{S-Q}{2} - \frac{S+Q}{2} = -Q \\
ad + bc &= \frac{S-Q}{2} + \frac{S+Q}{2} = S \\
ac - bd &= \frac{P-R}{2} - \frac{P+R}{2} = -R.
\end{aligned}
$$

Thus we see that

$$p = P^2 + Q^2 = S^2 + R^2 = (a^2 + b^2)(c^2 + d^2)$$

. Thus one of $a^2 + b^2$ or $c^2 + d^2$ must be 1, which means that one of a, b, c, d must be 0. Thus we know that $P^2 = R^2$ and $Q^2 = S^2$, which means that there is exactly one representation of p as a sum of two squares.