MATH 124 HOMEWORK #4

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(1) Suppose that $\left(\frac{13}{p}\right) = 1$. Since $13 \equiv 1 \pmod 4$, we know that $\left(\frac{p}{13}\right) = 1$. The quadratic residues modulo 13 are $1, 3, 4, -1, -3, -4$. Thus any prime congruent to any of these modulo 13 will have such a solution.

(2) (a) (i) First, consider $\langle 1, 1, \ldots \rangle$. This satisfies $\alpha = 1 + 1/\alpha$, or $\alpha^2 - \alpha - 1 = 0$. Thus we know that $\alpha = \frac{1+\sqrt{5}}{2}$.
So

$$\beta = \langle 2, 3, 1, 1, \ldots \rangle = 2 + \frac{1}{2 + (1 + \frac{1}{1+\cdots})} = 2 + \frac{1}{2+\alpha}.$$

Simplifying the right-hand side, we get

$$\beta = \frac{5 + 2\alpha}{2+\alpha} = \frac{5 + 2\frac{1+\sqrt{5}}{2}}{2 + \frac{1+\sqrt{5}}{2}} = \frac{12 + 2\sqrt{5}}{5 + \sqrt{5}}.$$

Rationalizing the denominator, we get

$$\beta = \frac{(12 + 2\sqrt{5})(5 - \sqrt{5})}{20} = \frac{25 - \sqrt{5}}{10}$$

(ii) First, we will evaluate $\alpha = \langle 1, 2, 1, 2, \ldots \rangle$. Then we know that

$$\alpha = 1 + \frac{1}{2 + \frac{1}{\alpha}}.$$

Simplifying this, we have

$$2\alpha^2 - 2\alpha - 1 = 0.$$

Then we have $\alpha = \frac{1+\sqrt{3}}{2}$.
But the

$$\beta = \langle 1, 3, 1, 2, 1, 2 \rangle = 1 + \frac{1}{3 + \frac{1}{\alpha}}.$$

Simplifying this, we get

$$\beta = 3 - \sqrt{3}.$$

(b) We claim that $\sqrt{5} = \langle 2, 4, 4, \ldots \rangle$. Setting $\alpha = \langle 4, 4, \ldots \rangle$ we know that $\alpha = 4 + 1/\alpha$. Solving this, we get $\alpha^2 - 4\alpha - 1 = 0$, so $\alpha = 2 + \sqrt{5}$. But then $\sqrt{5} = \alpha - 2 = \langle 2, 4, 4, \ldots \rangle$.
Notice that

$$\sqrt{20} = 2\sqrt{5} = 2\left(2 + \frac{1}{4 + \frac{1}{4+\cdots}}\right) = 4 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2+\cdots}}} = \langle 4, 2, 8, 2, 8, \ldots \rangle.$$

(On the even steps you multiply by 2 and on the odd ones you divide.)

(c) After looking at a few special cases, it becomes clear that the continued fraction for the number $\sqrt{n(n+1)}$ ought to be $\langle n, 2, 2n, 2, 2n, \ldots \rangle$.
First, consider $\alpha = \langle 2n, 2, 2n, 2, \ldots \rangle$. We get

$$\alpha = 2n + \frac{1}{2 + \frac{1}{\alpha}},$$

and solving it we get

$$\alpha = n + \sqrt{n(n+1)}.$$

Thus we know that $\sqrt{n(n+1)} = \alpha - n$ so the answer is $\langle n, 2, 2n, 2, 2n, \ldots \rangle$, as desired.

(3) In the proof of this problem we will assume that all numbers are positive. If one of the numbers is not, we add to both sides of the inequality until both numbers are positive.
Fact: Notice that if $\alpha = \langle a_1, a_2, a_3, \ldots \rangle$, then $1/(\alpha - a_1) = \langle a_2, a_3, \ldots \rangle$ (by definition). In particular, this means that $\langle a, a_2, a_3, \ldots \rangle < \langle a, b_2, b_3, \ldots \rangle$ if and only if $\langle a_2, a_3, \ldots \rangle > \langle b_2, b_3, \ldots \rangle$.

(a) Using the fact, we know that $c > d$ if and only if $\langle c \rangle > \langle d \rangle$, which happens if and only if $\langle a, c \rangle < \langle a, d \ldots \rangle$ for any positive a.
Similarly we can use the fact to add two numbers at the front. Then the sign will reverse twice, and so we will have $\langle a, b, c \langle > \langle a, b, d$.

(b) We assume that the a's are integers, but notice that the fact does not depend on c being an integer. In particular, we know that $a_n + c > a_n$. Thus we can use the fact n times (since we add n numbers to the front), to get

$$\langle a_0, \ldots, a_n + c \rangle ? \langle a_0, \ldots, a_n \rangle,$$

where ? is $>$ reversed $n-1$ times. In particular, if $n = 2k$, $? =>$ and if $n = 2k+1$, $? =<$, as desired.

(c) We assume that $a_n, b_{n+1} \neq 1$ (this can be accomplished by removing each and adding 1 to the previous number).
Suppose that k is the smallest integer such that $a_k \neq b_k$. We claim that $\langle a_0, a_1, \ldots \rangle < \langle b_0, b_1, \ldots \rangle$ if and only if $a_k ? b_k$, where ? is $<$ reversed k times.
Notice that if $a < b$, then $\langle a, \ldots \rangle < \langle b, \ldots \rangle$, since the rest of the fraction is strictly less than 1. This means that $\langle a_0, \ldots, a_{2\ell-1}, a, \ldots \rangle < \langle a_0, \ldots, a_{2\ell-1}, b, \ldots \rangle$ and $\langle a_0, \ldots, a_{2\ell}, a, \ldots \rangle > \langle a_0, \ldots, a_{2\ell}, b, \ldots \rangle$. Applying this to k above, we get the desired result.
Thus the condition we want is that if the two sequences differ for the first time at $a_{2\ell}$ and $b_{2\ell}$ then we want $a_{2\ell} < b_{2\ell}$, and if they differ for the first time at $a_{2\ell+1}$ and $b_{2\ell+1}$, then $a_{2\ell+1} > b_{2\ell+1}$.