MATH 124 HOMEWORK #12

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(1) (a) For $s > 1$,

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n\left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{sn}{u^{s+1}}\,du \\ &= s\int_1^{\infty} \frac{\lfloor u \rfloor}{u^{s+1}}\,du.\end{aligned}$$

(b) For $s > 1$,

$$\begin{aligned}\zeta(s) &= s\int_1^{\infty} \frac{\lfloor u \rfloor}{u^{s+1}}\,du = s\int_1^{\infty} \frac{u - \{u\}}{u^{s+1}}\,du \\ &= s\int_1^{\infty} \frac{1}{u^s}\,du - s\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\,du = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\,du,\end{aligned}$$

where splitting up the integral is valid as each part converges.

(c) Note that

$$\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\,du \le \int_1^{\infty} \frac{1}{u^{s+1}}\,du = -\frac{1}{s}u^{-s}\Big|_1^{\infty} = \frac{1}{s} < \infty$$

for $s > 0$.

(d) Suppose that $0 < s < 1$. Then

$$\int_0^1 \frac{\{u\}}{u^{s+1}}\,du = \int_0^1 \frac{u}{u^{s+1}}\,du = \int_0^1 \frac{1}{u^s}\,du = \frac{1}{1-s}u^{1-s}\Big|_0^1 = \frac{1}{1-s}.$$

Thus

$$-s\int_0^{\infty} \frac{\{u\}}{u^{s+1}}\,du = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\,du = \zeta(s),$$

as desired.

Note that the integral on the left above is positive, as $u > 0$. Thus the left-hand side of the equality is negative, and therefore $\zeta(s) < 0$ when $0 < s < 1$.

(2) (a) We use an analogous proof to that of theorem 8.24.

If we let $F(n) = \sigma(n)/n$, we have $F(n) = \sum_{d|n} \frac{d}{n}$, by definition. But then we can take $f(d) = \frac{1}{d}$, and we will have $F(n) = \sum_{d|n} f(d)$. Applying equation (8.48), we see that

$$\sum_{n \le x} F(n) = x\sum_{d \le x} \frac{f(d)}{d} + O\left(\sum_{d \le x} |f(d)|\right).$$

In particular, we know that $f(d)/d = 1/d^2$, so the first summation is

$$\zeta(2) - \sum_{d>x} \frac{1}{d^2}.$$

Note that we know that

$$\sum_{d\leq x} \frac{1}{d} = O(\log x).$$

So the equation above gives us

$$\begin{aligned}\sum_{n\leq x} \frac{\sigma(n)}{n} &= x\left(\zeta(2) - \sum_{d>x}\frac{1}{d^2}\right) + O(\log x) = x\zeta(2) + O(\log x) \\ &= x\frac{\pi^2}{6} + O(\log x),\end{aligned}$$

where we note that

$$x\sum_{d>x}\frac{1}{d^2} = xO(1/x) = O(\log x).$$

(b) Defining f_k as in the last problem set, we note that

$$f_k(n) = \sum_{d^k|n} \mu(d)$$

(as if we write $n = rs^k$, with r k-th power free, we have $f(n) = 1$ iff $s = 1$, which means if and only if $\sum_{d|s}\mu(d) = 1$). But then if we apply (8.48) (analogously to the application in theorem 8.25) we get

$$\sum_{n\leq x} f_k(n) = x\sum_{d\leq x^{1/k}} \frac{\mu(d)}{d^k} + O\left(\sum_{d\leq x^{1/k}} \mu(d)\right).$$

The second sum above is at most $x^{1/k}$. Considering the first sum, we see

$$\sum_{d\leq x^{1/k}} \frac{\mu(d)}{d^k} = \sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} - \sum_{d>x^{1/k}}\frac{\mu(d)}{d^k}.$$

Note that the second sum on the right-hand side is at most $x^{\frac{-(k-1)}{k}}$, so we get

$$\sum_{n\leq x} f_k(n) = x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} - xO(x^{-\frac{k-1}{k}}) + O(x^{1/k}) = x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + O(x^{1/k}).$$

Now all we need to do is compute the first sum. But we know that $M(s)\zeta(s) = 1$ (where M is the Dirichlet series associated to μ) so we get that $M(k) = 1/\zeta(k)$, turning the sum into the desired form.

(3) (a) Notice that

$$\begin{aligned}
\frac{D(x)}{x}+\int_1^x \frac{D(u)}{u^2}\,du &= \sum_{n\le x}\frac{d(n)}{x}+\sum_{n\le x}\int_n^{\min(x,n+1)}\frac{D(n)}{u^2}\,du \\
&= \sum_{n\le x}\frac{d(n)}{x}+\sum_{n\le x}\left(\sum_{m\le n} d(m)\right)\left(\frac{1}{n}-\frac{1}{\min(x,n+1)}\right) \\
&= \sum_{n\le x}\frac{d(n)}{x}+\sum_{m\le x} d(m)\sum_{m\le n\le x}\left(\frac{1}{n}-\frac{1}{\min(x,n+1)}\right) \\
&= \sum_{n\le x}\left(\frac{d(n)}{x}+d(n)\left(\frac{1}{n}-\frac{1}{x}\right)\right) = \sum_{n\le x}\frac{d(n)}{n}
\end{aligned}$$

as desired.

(b) From theorem 8.28 we know that $D(x) = x\log x + (2\gamma-1)x + O(\sqrt{x})$. In particular, this means that

$$\begin{aligned}
\int_1^x \frac{D(u)}{u^2}\,du &= \int_1^x \frac{u\log u + 2(\gamma-1)u + O(\sqrt{u})}{u^2}\,du \\
&= \int_1^x \frac{\log u}{u}\,du + \int_1^x (2(\gamma-1)u^{-1}+u^{-3/2})\,du \\
&= \frac{1}{2}(\log x)^2 + O(\log x).
\end{aligned}$$

In addition, we know that

$$\frac{D(x)}{x} = \log x + (2\gamma-1) + O(x^{-1/2}) = O(\log x).$$

Plugging these in to part (a) we see

$$\sum_{n\le x}\frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + O(\log x)$$

as desired.

(c) Recall from the previous problem set that $d_3(n) = \sum_{k|n} d(n)$. Therefore we see that

$$\begin{aligned}
\sum_{n\le x} d_3(n) &= \sum_{n\le x}\sum_{k|n} d(k) = \sum_{k\le x}\left\lfloor\frac{x}{k}\right\rfloor d(k) \\
&= x\sum_{k\le x}\frac{d(k)}{k} - \sum_{k\le x}\left\{\frac{x}{k}\right\} d(k) = x\sum_{k\le x}\frac{d(k)}{k} - O(D(x)).
\end{aligned}$$

Plugging in the equation from theorem 8.28 and the result of part (b) we see

$$\sum_{n\le x} d_3(n) = \frac{1}{2}x(\log x)^2 + xO(\log x) - O(x\log x) = \frac{1}{2}x(\log x)^2 + O(x\log x).$$