MATH 124 HOMEWORK #11

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(1) (a) To show that this diverges it suffices to show that the terms do not converge to 0. We know that, letting $k = \lceil s \rceil$, $\frac{2^n}{n^s} \geq \frac{2^n}{n^k}$, so it suffices to show that $\frac{2^n}{n^k} \not\to 0$. But, after differentiating top and bottom k times by L'Hopital we have

$$\lim_{n\to\infty} \frac{2^n}{n^k} = \lim_{n\to\infty} \frac{\log^k 2 2^n}{k!} = \infty$$

as desired. So each term does not tend to 0, so the sum diverges for all k.

(b) Notice that for $s \geq 0$ we have $\frac{1}{2^n n^s} \leq \frac{1}{2^n}$, and the sum $\sum_{n=1}^{\infty} \frac{1}{2^n} = 2 < \infty$, so for $s > 0$ this sum converges.

Now it remains to show that for $s > 0$ the sum $\sum \frac{n^s}{2^n}$ converges. But, from the integral test,

$$\begin{aligned} \sum \frac{n^s}{2^n} &\leq \int_1^{\infty} x^s 2^{-x}\, dx \\ &< \int_1^{\infty} x^k e^{-x}\, dx \\ &= \sum_{j=0}^{k} k(k-1)\cdots(k-j+1)x^{k-j}e^{-x}\Big|_1^{\infty} \\ &= \sum_{j=0}^{k} k(k-1)\cdots(k-j+1) < \infty \end{aligned}$$

so the sum converges, as desired.

(c) Notice that

$$\sum_{n=1}^{\infty} \frac{1}{ny} = \frac{1}{y}\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Thus the left-hand side of the equation does not exist. On the other hand,

$$\lim_{y\to\infty} \frac{1}{yn} = 0,$$

so the right-hand side is always equal to zero.

The hypothesis of dominated convergence which is never satisfied is that the sum $\sum_{n=1}^{\infty} M_n$ converges. Notice that we must have $M_n \geq \frac{1}{an}$ for each n.

(2) For any function $a(n)$ we will write $A(n) = \sum_{n=1}^{\infty} a(n)n^{-s}$. We write id for the identity function. In this problem, we will assume that s is large enought that all series converge.

First, we want to show that $\mu * d = 1$. In particular, this means that we need to show that $M(s)D(s) = 1$. We know that $D(s) = \zeta(s)^2$, and that

$$M(s) = \prod_p \left(1 - \frac{1}{p^{-s}}\right) = \frac{1}{\zeta(s)}.$$

Thus

$$M(s)D(s) = \zeta(s)$$

implying that $\mu * d(n) = 1$, as desired.

Second, we want to show that $\mu*\sigma = \mathrm{id}$. Notice that $\mathrm{id}(s) = \zeta(s-1)$. From above, we know that $M(s) = \zeta(s)^{-1}$. $\sigma(n) = \mathrm{id} * 1(n)$, so we have $\Sigma(s) = \zeta(s-1)\zeta(s)$. Thus

$$M(s)\Sigma(s) = \zeta^{-1}(s)\zeta(s-1)\zeta(s) = \zeta(s-1) = \mathrm{id}(s).$$

Thus $\mu * \sigma = \mathrm{id}$, as desired.

Lastly, we want to show that

$$\sigma * 1(n) = \sum_{k|n} \sigma(k) \stackrel{?}{=} n \sum_{k|n} \frac{d(k)}{k} = \sum_{k|n} d(k)\frac{n}{k} = d * \mathrm{id}(n).$$

The left-hand side of this expression is $\Sigma(s)\zeta(s) = \zeta(s-1)\zeta(s)^2$. The right-hand side of this is $D(s)\zeta(s-1) = \zeta(s)^2\zeta(s-1)$, as desired.

(3) (a) Notice that

$$\sum \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{-s}} + \cdot + \frac{1}{p^{-(k-1)s}}\right)$$

as the k-th power free numbers are exactly those that are not divisible by p^k for any p. But

$$\prod_p \left(1 + \frac{1}{p^s} + \cdot + \frac{1}{p^{(k-1)s}}\right) = \prod_p \frac{1 - p^{-ks}}{1 - p^{-s}} = \frac{\zeta(s)}{\zeta(ks)}$$

as desired.

(b) (i) Notice that the number of pairs (x, y) such that $xy = n$ is exactly the same as the number of choices of x such that $x|n$ (since y must then be $\frac{n}{x}$), which means that it is the number of divisors of n, $d(n)$.

(ii) Consider

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^k.$$

The coefficient of n^{-s} will be the number of ways that n could be written as a product of k divisors, as we have to choose one from each of the copies of the sum that we are multiplying. So

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

On the other hand, this is exactly $\zeta(s)^k$, so we have the desired equality.

(4) By theorem 8.13, if we can find the greatest lower bound for c such that $f(y) = O(x^c)$, then we will have $\sigma = c$.

Consider $y = 10^k$. We can consider all integers less than y to be k-tuples of digits. (We can pad shorter numbers and it won't matter, as the padding is not 9's.) But all k-tuples of digits appear as such numbers, so we know exactly how many there are: there are 9^k tuples with no 9's. Thus

$$f(10^k) = 9^k,$$

so we can write for $x = 10^k$

$$f(x) = 9^{\log_{10} x} = x^{\log_{10} 9}.$$

Thus we know that for any $c < \log_{10} 9$, $f(x) = \omega(x^c)$, since for any fixed constant A there will be a k such that for $x = 10^k$

$$x^{\log_{10} 9} > Ax^c.$$

On the other hand, we know that if x has first digit 9 and $k+1$ digits, $f(x) = f(9 \cdot 10^k - 1)$; also, if x has a first digit $1 \leq \ell \leq 8$, $f(x) = \ell f(10^k) + f(x - 10^k)$. Thus we know that $f(x) = O(f(10^k))$, so we know that for all x,

$$f(x) = O(x^{\log_{10} 9})$$

and that $c = \log_{10} 9$ is the minimum such c; thus we are done.