MATH 250 HOMEWORK #1

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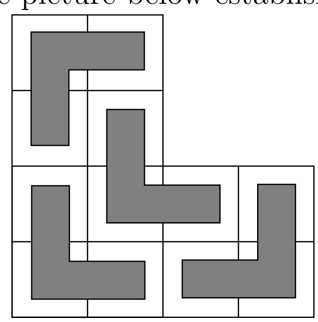
(1) (a) We will prove this by induction on n. Notice that when $n = 1$ we have only a triomi no to fill, so we can clearly tile the square with tripminoes. Now suppose that we can tile any $2^{n-1} \times 2^{n-1}$ square (with one square removed) with triominoes. We will show that this means that we can tile any $2^n \times 2^n$ square (with one square removed) with triominoes.

Split the $2^n \times 2^n$ square into 2×2 blocks in hte obvious fashion, and remove the 2×2 block containing the removed square. Now we have a $2^{n-1} \times 2^{n-1}$ square composed of 2×2 blocks. We can tile this with "doubled" triominoes (where we replace each square of a triomino with a 2×2 block). Now, if we can show that we can tile a "doubled" triomino with triominos, and that we can tile the removed squares with triominoes, we will be done. We can clearly tile the removed squares with a triomino, since the removed squares are simply a 2×2 block with one square removed, and we have already shown that we can tile that with triominos. So it remains to show that it is possible to tile a "doubled" triomino with triominos; the picture below establishes that this is possible:



(b) We will prove this by induction on n. Note that when $n = 1$ this is trivial, since it reduces to $1^3 = (1)^2$, which is obvious. Now suppose that we have shown this to be the case for $n - 1$; we will prove it for n. Note that

$$(1 + 2 + \cdots + n)^2 = \frac{n^2(n+1)^2}{4}$$

from the summation formula for integers. But from the induction hypothesis we have

$$\begin{aligned} 1^3 + 2^3 + \cdots + (n-1)^3 + n^3 &= \frac{(n-1)^2 n^2}{4} + n^3 \\ &= \frac{n^4 - 2n^3 + n^2 + 4n^3}{4} \\ &= \frac{n^2(n+1)^2}{4}, \end{aligned}$$

as desired.

(c) Note that for any divisor $d < n$, there is another divisor of n which is n/d. If $d \leq \sqrt{n}$, then $n/d \geq \sqrt{n}$ and vice versa. Thus if n is not a perfect square we can write

$$\prod_{d|n} d = \prod_{d|n,\ d<\sqrt{n}} d \cdot \frac{n}{d} = n^{\#\{d\,:\,d|n, d<\sqrt{n}\}}.$$

However, because we have paired up the divisors and only include one of each pair in the set, this is exactly $n^{d(n)/2}$.

Now suppose that n is a perfect square. The entire analysis above appliees if we multiply everything past the first step by $\sqrt{n}$ and $d(n)$ by $d(n)-1$ (since we remove the one divisor $\sqrt{n}$ from the product). Thus we get that

$$\prod_{d|n} d = n^{(d(n)-1)/2}\sqrt{n} = n^{d(n)/2}$$

as desired.

(2) (a)

$$\begin{aligned} 7469 &= 2464 \cdot 3 + 77 \\ 2464 &= 77 \cdot 32. \end{aligned}$$

Thus we have that

$$77 = 7469 - 2464 \cdot 3.$$

(b)

$$\begin{aligned} 4999 &= 1109 \cdot 4 + 563 \\ 1109 &= 563 \cdot 1 + 546 \\ 563 &= 546 \cdot 1 + 17 \\ 546 &= 17 \cdot 32 + 2 \\ 17 &= 2 \cdot 8 + 1 \\ 2 &= 1 \cdot 2. \end{aligned}$$

Thus we have

$$\begin{aligned} 1 &= 17 - 2 \cdot 8 \\ &= 17 - (546 - 17 \cdot 32) \cdot 8 = 17 \cdot 257 - 546 \cdot 8 \\ &= (563 - 546 \cdot 1) \cdot 257 - 546 \cdot 8 = 563 \cdot 257 - 546 \cdot 265 \\ &= 563 \cdot 257 - (1109 - 563 \cdot 1) \cdot 265 = 563 \cdot 522 - 1109 \cdot 265 \\ &= (4999 - 1109 \cdot 4) \cdot 522 - 1109 \cdot 265 = 4999 \cdot 522 - 1109 \cdot 2353. \end{aligned}$$

(3) (a) If $x = a + b\sqrt{-5}$ we will let $\bar{x} = a - b\sqrt{-5}$. Note that $\bar{xy} = \bar{x} \cdot \bar{y}$ (from a similar argument to that of complex multiplication). Then we know that

$$N(xy) = (xy)(\bar{xy}) = xy\bar{x}\bar{y} = x\bar{x}y\bar{y} = N(x)N(y).$$

(b) Notice that a number x is a unit if and only if $N(x) = 1$. If $N(x) > 1$ then there cannot exist y such that $N(xy) = 1$, since $N(xy) = N(x)N(y)$, adn $N(y) \geq 1$ and $N(x) > 1$. If $N(x) = 1$ then $x\bar{x} = 1$, so x has an inverse (namely $\bar{x}$) and is therefore is a unit.

Thus we want to find all x such that $N(x) = 1$, or all pairs of integers (a, b) such that $a^2 + 5b^2 = 1$. However, this is clearly only possible if $a^2 = 1$, so the pairs of integers are $(-1, 0)$ and $(1, 0)$ and the units are 1 and -1.

(c) Suppose that $2 = wz$. Then we know that $4 = N(2) = N(w)N(z)$. If both w and z are not units then they both have norm greater than 1, and thus must both have norm 2. However, since 2 is neither a square nor a sum of the form $a^2 + 5b^2$ this is impossible, so 2 is irreducible.

(d) Suppose $1 + \sqrt{-5} = wz$. Then $6 = N(1 + \sqrt{-5}) = N(w)N(z)$. If both w and z are not units then their norms must be 2 and 3. However, with the same argument as in the previous part, we can see that this is impossible. Therefore $1 + \sqrt{-5}$ is irreducible.

(e) Notice that the above two arguments also show that 3 and $1 - \sqrt{-5}$ are irreducible. (Note that if x is irreducible then so is $\bar{x}$, since if we can write $x = wz$ with $N(w), N(z) > 1$, then we can write $\bar{x} = \bar{w} \cdot \bar{z}$, with $N(\bar{w}) = N(w) > 1$ and $N(\bar{z}) = N(z) > 1$.) But then we have two factorizations of 6:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

(4) (a) If n is not prime it has a proper divisor d greater than 1, and thus $\sigma(n) \geq 1 + d + n > 1 + n$, so $\sigma(n) \neq n + 1$. If n is prime then its only divisors are 1 and itself, so $\sigma(n) = n + 1$. Thus $\sigma(n) = n + 1$ if and only if n is prime.

(b) Suppose that $\sigma(n) = n + k$, where $k|n$ and $k < n$. Suppose that $k > 1$ (so n is not prime). Then we know three distinct divisors of n: $n, k, 1$, and thus $n + k = \sigma(n) \geq n + k + 1$, a contradiction. Thus $k = 1$ and so by the previous part, n is prime.

(c) Suppose that n is an even perfect number. We can write $n = 2^k m$, where $k \geq 1$ and m is odd. We can partition the divisors of n into $k + 1$ sets S_i, where each divisor in S_i is divisible by 2^i. Notice that $\sum_{a \in S_i} a = 2^i \sigma(m)$, since each S_i contains each divisor of m multiplied by 2^i. Thus we know that

$$2^{k+1} m = \sigma(n) = \sum_{i=0}^{k} \sum_{a \in S_i} a = \sum_{i=0}^{k} 2^i \sigma(m) = (2^{k+1} - 1)\sigma(m). \tag{1}$$

So we have

$$2^{k+1}(\sigma(m) - m) = \sigma(m).$$

Write $\sigma(m) = m + \ell$. Then we have

$$2^{k+1} \ell = m + \ell. \tag{2}$$

Since ℓ divides the left-hand side of the expression and one part of the right-hand side, we know that it must also divide m. Also, notice that $m + \ell = 2^{k+1}\ell \geq 4\ell$, which implies that $m > \ell$. Applying part (b), we see that m must be prime, and $\ell = 1$. From equation (2) we conclude that $m = 2^{k+1} - 1$ and is prime, so $n = 2^k(2^{k+1} - 1)$ with $2^{k+1} - 1$ prime.