Midterm

Math 124, Fall 2005

November 4, 2005

Good luck!

1. (a) **Show that $\sqrt{18}$ is representable by the continued fraction.**

$$\sqrt{18} =< 4, 4, 8, 4, 8, 4, 8 \cdots >$$

Say $\theta =< 8, 4, 8, 4, \cdots >$. We have that

$$\theta = 8 + \frac{1}{4 + \frac{1}{\theta}} = 8 + \frac{\theta}{4\theta + 1} = \frac{32\theta + 8 + \theta}{4\theta + 1}$$

Hence

$$4\theta^2 + \theta = 33\theta + 8 \qquad \Longrightarrow \qquad \theta^2 - 8\theta - 2 = 0.$$

In particular,

$$\theta = \frac{8 \pm \sqrt{64 + 8}}{2} = 4 + \sqrt{18}$$

Since $\theta > 4$, we get that

$$\sqrt{18} =< 4, 4, 8, 4, 8, 4, 8 \cdots >$$

(b) **Find a solution to the equation $x^2 - 18y^2 = 1$.** The convergents of the continuous fraction for $\sqrt{18}$ gives

$$\frac{A_1}{B_1} = 4 + \frac{1}{4} = \frac{17}{4}$$

Hence $x = 17, y = 4$ gives a solution.

2. **Assume that we have an integral solution to the equation $y^2 = x^3 + 7$.**

(a) **Assume that x is even. Get a contradiction by considering the equation modulo 8.**

If $x = 2k$ then

$$y^2 = (2k)^3 + 7 \equiv 7 \mod 8.$$

But the only quadratic residues modulo 8 are 1 and 4. Hence x cannot be even.

(b) **Show that if x is odd then $x^2 - 2x + 4$ has a prime factor q of the form $4k + 3$.**

If $x = 2k + 1$ then

$$A = x^2 - 2x + 4 \equiv (2k + 1)^2 - 2(2k + 1) \equiv 1 - 2 \equiv 3 \mod 4$$

Assume that all the primes factors of A are of the form $4k + 1$ then we would have

$$A = \prod (4k_i + 1) \equiv \prod 1 \equiv 1 \mod 4$$

which is false. We therefore know that A has at least one prime factor $q = 4k + 1$.

(c) **Prove that there are no solutions. [Hint : Use that $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$ and consider the equation modulo q.]**

$$y^2 + 1 = (x + 2)(x^2 - 2x + 4) \equiv 0 \mod q$$

Hence -1 is a quadratic residue modulo $q = 4k + 3$ which we know is impossible.

3. (a) **Say a form $ax^2 + bxy + cy^2$ has discriminant -16. Write c in terms of a and b. What is b congruent to modulo 4.**

$$-16 = b^2 - 4ac \implies c = \frac{b^2 + 16}{4a}$$

Hence 4 must divide b^2 and so

$$b \equiv 0 \mod 2.$$

(b) **Find all reduced forms of discriminant -16.**

For (a, b, c) to be reduced, we first need that

$$|b| \leq a \leq c \implies b^2 \leq ac \implies -16 \leq -3ac \text{ or } ac \leq 5.$$

Hence a must be either 1 or 2. Remember that b must be even and $|b| \leq a$.

$$\begin{aligned} a = 1 &\implies b = 0 \ \& \ c = 4 \\ a = 2 &\implies b = 0 \ \& \ c = 2 \\ &\quad \text{or } b = 2 \ \& \ c = 5/2 \quad \text{not valid} \end{aligned}$$

Hence the only reduced quadratic forms of discriminant -16 are

$$x^2 + 4c^2 \qquad \& \qquad 2x^2 + 2y^2$$

(c) **For each such form, find an integer represented by it but not by the others. [Consider small positive integers.]**

The form $x^2 + 4y^2$ represents 1 while $2x^2 + 2y^2$ is always at least 2. Also $2x^2 + 2y^2$ represents 2 while $x^2 + 4y^2$ does not represent any integer between 1 and 4.

(d) **From 3c, conclude briefly that these forms are not equivalent.**

Equivalent forms represent equivalent numbers.

Bonus. Find all solutions to the system of equations

$$\begin{aligned} x &\equiv 2 \mod 15 \\ x &\equiv 11 \mod 21 \end{aligned}$$

Both 15 and 21 are divisible by 3. We therefore need to check that these equations are compatible modulo 3. Both reduce to

$$x \equiv 2 \mod 3.$$

Hence, we can find a solution and this solution is unique modulo $15 \times 7 = 105$. We need to solve the following system.

$$\begin{aligned} x &\equiv 2 & \mod 15 \\ x &\equiv 11 \equiv 4 & \mod 7 \end{aligned}$$

Using that

$$\begin{aligned} (-2) \times 7 &\equiv 1 \mod 15 \\ 15 &\equiv 1 \mod 7 \end{aligned}$$

we get that

$$x = 15 \times 4 + -14 \times 2 = 60 - 28 = 32$$

is a solution. Any other solutions needs to be congruent to 32 modulo 105. Hence

$$32 + k105$$

give all possible solutions.