Homework 5

Math 124, Fall 2005

Due Wednesday, October 26

No late assignments will be accepted as solutions will be posted on Thursday morning. Solve three of the following four problems. (The fourth one will be considered a bonus.)

Throughout this assignment we will denote by

$$< q_0, q_1, \dots q_n >= q_0 + \cfrac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \dots}}}$$

1. (a) Find the irrational number α having continued fraction expansion

$$< 4, 4, 8, 4, 8, 4, 8, \cdots > .$$

   (b) For each of the following equation find the smallest possible pair of integers (x, y) which solve it. (If they exists.)

$$\begin{aligned} x^2 - \alpha^2 y^2 &= -1 \\ x^2 - \alpha^2 y^2 &= 1 \end{aligned}$$

2. Here d is any integer which is not a perfect square. Assume that the smallest positive solution to the equation

$$x^2 - dy^2 = 1 \tag{1}$$

   is (a, b).

   (a) Show that for any n, the pair x_n and y_n determined by

$$x + \sqrt{d}y = (a + b\sqrt{d})^n$$

   are also solutions to (1).

   (b) Prove that any positive solution is of this form. Assume there is a positive solution (s, t) to (1) which is not equal to any of (x_n, y_n). Let m be the integer such that

$$(a + b\sqrt{d})^m < s + t\sqrt{d} < (a + b\sqrt{d})^{m+1}.$$

   i. Show that the u and v given by

$$1 < u + v\sqrt{d} = \frac{s + t\sqrt{d}}{(a + b\sqrt{d})^m} < (a + b\sqrt{d})$$

   are integers.

ii. Show that (u, v) gives an integral solution to (1).

iii. Show that $(u + v\sqrt{d})^{-1} > 0$ and use this to show that both u and v are positive.

iv. Find a contradiction with the minimality of (a, b).

3. (a) Prove that the sum of the first n natural numbers is a perfect square for infinitely many values of n. [Hint: You may need that $4n^2 + 4n = (2n + 1)^2 - 1$.]

   (b) An integer is triangular if it can be expressed as $k(k + 1)/2$. Find all perfect squares that are triangular.

4. (a) Find two ways of expressing

$$(a^2 + b^2)(c^2 + d^2)$$

   as a sum of two squares. Conclude that, in general, such a product is expressible as the sum of two squares in at least two different ways.

   (b) Show that a prime which is the sum of two squares can only be expressed in one way. [Hint : Let

$$p = P^2 + Q^2 = R^2 + S^2$$

   and assume that P and R are even and S and Q are odd. Use unique factorization to choose integers a, b, c and d

$$P^2 - R^2 = S^2 - Q^2 = 4abcd.$$

   Prove that one of a,b,c or d is equal to 0.