Homework 11

Math 124, Fall 2005

Due Wednesday, Decmber 7th

No late assignments will be accepted.

1. (a) Show that the Dirichlet series
$$A(s) = \sum \frac{2^n}{n^s}$$
diverges for all s.

   (b) Show that the Dirichlet series
$$B(s) = \sum \frac{1}{2^n n^s}$$
converges for all s.

   (c) Consider the functions
$$f_n(y) = \frac{1}{yn}.$$
Show that
$$\lim_{y\to\infty} \sum_n f_n(y) \neq \sum_n \lim_{y\to\infty} f_n(y).$$
Which hypothesis of dominated convergence could never be satisfied?

2. Use Dirichlet series to prove the following identities for all integers n.
$$\begin{aligned} \sum_{k|n} \mu(k)d(n/k) &= 1 \\ \sum_{k|n} \mu(k)\sigma(n/k) &= n \\ \sum_{k|n} \sigma(k) &= n \sum_{k|n} d(k)/k \end{aligned}$$

3. (a) We say that a positive integer n is kth power free if 1 is the largest kth power that divides n. Define
$$f(n) = \begin{cases} 1 & n \text{ is } k\text{th power free} \\ 0 & \text{otherwise} \end{cases}$$
and show that for $s > 1$
$$\sum \frac{f(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$$

(b) Let $d_k(n)$ denote the number of ordered k-uples $(d_1, \ldots, d_k)$ of positive integers such that

$$d_1 \ldots d_k = n.$$

i. Show that $d_2(n) = d(n)$.

ii. Show that for $s > 1$ and for any $k \geq 1$

$$\sum \frac{d_k(n)}{n^s} = \zeta^k(s).$$

4. Let $\mathcal{N}$ denote the set of positive integers n whose base 10 representation does not contain the digit 9. Find the abscissa of absolute convergence of the Dirichlet series

$$\sum_{n \in \mathcal{N}} \frac{1}{n^s}.$$

[Hint : Find the least c for which the function

$$f(y) = \#\{n \leq y,\, n \in \mathcal{N}\}$$

is of the order x^c.]