

Homework 1

Math 124, Fall 2005

Due Wednesday, Sept 28th

Good luck!!

1. Prove the following statements.

(a) Show that any $2^n \times 2^n$ board with one square deleted can be covered by "triminoes" (see the figure).

(b) For any $n = 1, 2, \dots$,

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

(c) For any $n = 1, 2, \dots$,

$$\prod_{d|n} d = n^{\frac{d(n)}{2}}$$

2. By using the Euclidean algorithm, find the highest common factor h of the following numbers n and m . Use your work to find integers x and y so that $xn + ym = h$.

(a) 7469 and 2464

(b) 1109 and 4999.

3. Consider the ring $R = \mathbf{Z}[\sqrt{-5}]$. Its elements have the form $a + b\sqrt{-5}$ where both a and b are integers in \mathbf{Z} . The multiplication is given by

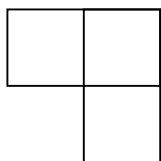
$$(a_1 + b_1\sqrt{-5})(a_2 + b_2\sqrt{-5}) = (a_1a_2 - 5b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{-5}.$$

Define the norm of $a + b\sqrt{-5}$ to be

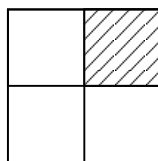
$$N(a + b\sqrt{-5}) = (a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2.$$

(a) Prove that if $x, y \in R$ then $N(xy) = N(x)N(y)$.

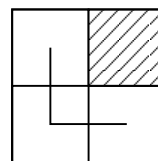
(b) Find the units in R , meaning the divisors of 1.



(a) A triminoe



(b) Filling a 2×2 board



- (c) Show that 2 is irreducible¹ in R , meaning that if $2 = wz$ then one of w and z is a unit.
- (d) Show that $(1 + \sqrt{-5})$ is irreducible in $\mathbf{Z}[\sqrt{-5}]$.
- (e) Show that 6 has two different factorizations into irreducibles.
4. Recall that $\sigma(n)$ is the sum of the divisors of n and prove the following statements.
- (a) Prove that an integer n is prime if and only if $\sigma(n) = n + 1$.
- (b) Prove that if $\sigma(n) = n + k$ where $k|n$ and $k < n$ then n is prime.
- (c) Recall that a number is perfect if $\sigma(n)$ is equal to $2n$. Show that if n is an even perfect number then $n = 2^{k-1}p$ where p is prime and equal to $2^k - 1$. (We have already shown in class that any number built in this way is perfect. It is not known if there are any odd perfect numbers.)

¹Let me explain why I am using the term *irreducible* instead of *prime*. In ring theory, a number n is prime if

$$n|(ab) \implies n|a \text{ or } n|b$$

while n is irreducible if whenever we have $n = ab$ then exactly one of a and b is a unit. Although these concepts agree in the integers, they differ in this case. For example, 2 is irreducible in R but not prime.