Math 124 Homework 6 Solutions

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1. For $n = 0$, the continued fraction is simply 1. For $n < 0$, the continued fraction is the same as for $-n$, so we only need to consider the case $n > 0$.

The first term is the greatest integer less than $\sqrt{n^2+1}$, which is n. The second term is the greatest integer less than

$$\frac{1}{\sqrt{n^2+1}-n} = \sqrt{n^2+1}+n$$

which is $2n$. The third term is then the greatest integer less than

$$\frac{1}{\sqrt{n^2+1}+n-2n} = \frac{1}{\sqrt{n^2+1}-n}$$

By the previous computation, it is again $2n$. Moreover, we are left with the same remainder, so the fourth and following terms will similarly be $2n$.

So, the continued fraction for $n > 0$ is

$$\sqrt{n^2+1} = n + \cfrac{1}{2n+\frac{1}{2n+\frac{1}{2n+\cdots}}}$$

$$= [n, 2n, 2n, 2n, \ldots]$$

2a. $n+1$ and $2n+1$ are perfect squares if and only if $n+1 = x^2$ and $2n+1 = y^2$ for some $x, y \in \mathbb{Z}$. Eliminating n, we find $y^2 - 2x^2 = -1$. This means that the number $y + x\sqrt{2}$ has norm -1 in $\mathbb{Q}[\sqrt{2}]$. Then, $(y+x\sqrt{2})^{2k+1}$ has norm $(-1)^{2k+1} = -1$ for $k \in \mathbb{N}$. So, if $a + b\sqrt{2} = (y + x\sqrt{2})^{2k+1}$, then $a^2 - 2b^2 = -1$ gives another solution.

Hence, it suffices to find a solution to $y^2 - 2x^2 = -1$ such that $(y + x\sqrt{2})^{2k+1}$ takes on infinitely many values. For example, we may take $x = 1$, $y = 1$, and then we obtain infinitely many solutions from $(1+\sqrt{2})^{2k+1}$. Each solution of this gives a solution for n.

2b. Plugging in values of k and doing the calculations, we find the solutions $n = 840$ and $n = 28560$ when $k = 2, 3$.

3a. The first two conditions are already satisfied by the f given by the algebraic property of α. We will use the following two processes to obtain an f that satisfies the third and fourth properties:

(1) If f has a rational root p/q, then we can construct g such that g also satisfies the first two conditions, p/q is not a root of g, and g has degree less than f. Simply divide $f(x)$ by $x - p/q$; by looking at how synthetic division works, for example, it is easy to see that the resulting polynomial has rational coefficients. Then, one can multiply by an integer to clear the denominators and be left with a polynomial with integer coefficients. α would still be a root of f since $f(x)$ would still have a factor of $x - \alpha$. One may continue dividing by $x - p/q$ until f has no factors left, so then p/q is not a root. Clearly the resulting polynomial has degree less than that of f.

(2) If $f'(\alpha) = 0$, then we can construct g such that g also satisfies the first two conditions, $g'(\alpha) \neq 0$, and g has degree less than f. Consider $f'(x)$. Then, it satisfies the first condition $f'(\alpha) = 0$, and f' also has integer coefficients. Similarly, every derivative of f satisfies the first two conditions. Note that the nth derivative of f is $n!a_n$, so $f^{(n)}(\alpha) = n!a_n \neq 0$. Hence, there exists an m such that m is the least positive integer such that $f^{(m)}(\alpha) \neq 0$. Set $g = f^{(m-1)}$. Then $g(\alpha) = 0$ and $g'(\alpha) \neq 0$ by how m was defined. Clearly g has a smaller degree than f. Hence we have found our desired g.

Now, given any f satisfying the first two properties, apply (1) if f has a rational root, and apply (2) if f does not satisfy $f'(\alpha) \neq 0$. This gives a new polynomial of smaller degree also satisfying the first two properties. Any polynomial with integer coefficients having α as a root must have degree at least 2, because a polynomial of degree 1 $a_1x + a_0 = 0$ has only the rational solution $x = -a_1/a_0$. Therefore, this process must terminate, since applying (1) or (2) retains the property that α is a root and it has integer coefficients. However, if the process terminates, that means neither (1) nor (2) may be applied. That is, f has no rational root and $f'(\alpha) \neq 0$. That proves that a desired f exists.

3b. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, where the a_i are integers. We have

$$\begin{aligned}\left|f\left(\frac{p}{q}\right)\right| &= \left|a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_0\right| \\ &= \frac{|a_np^n + a_{n-1}p^{n-1}q + \cdots + a_0q^n|}{q^n}\end{aligned}$$

Since the numerator is an integer and not 0 (otherwise f would have a rational root), it must be at least one. That gives the inequality

$$\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^n}$$

3c. Use Taylor's theorem applied to f around α. That gives

$$f\left(\frac{p}{q}\right) = f(\alpha) + \frac{f'(\alpha)}{1!}\left(\alpha - \frac{p}{q}\right) + \frac{f''(\alpha)}{2!}\left(\alpha - \frac{p}{q}\right)^2 + \cdots$$

Substitute $f(\alpha) = 0$ and take absolute values:

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{f'(\alpha)}{1!}\left(\alpha - \frac{p}{q}\right) + \frac{f''(\alpha)}{2!}\left(\alpha - \frac{p}{q}\right)^2 + \cdots\right|$$

By the triangle inequality, we have

$$\left|\frac{f'(\alpha)}{1!}\left(\alpha - \frac{p}{q}\right)\right| - \left|\frac{f''(\alpha)}{2!}\left(\alpha - \frac{p}{q}\right)^2\right| - \cdots \le \left|f\left(\frac{p}{q}\right)\right|$$

$$\le \left|\frac{f'(\alpha)}{1!}\left(\alpha - \frac{p}{q}\right)\right| + \left|\frac{f''(\alpha)}{2!}\left(\alpha - \frac{p}{q}\right)^2\right| + \cdots$$

Substitute $|\alpha - p/q| = \epsilon$ to get

$$\epsilon|f'(\alpha)| - \epsilon^2\frac{|f''(\alpha)|}{2!} - \cdots \le \left|f\left(\frac{p}{q}\right)\right| \le \epsilon|f'(\alpha)| + \epsilon^2\frac{|f''(\alpha)|}{2!} + \cdots$$

As ϵ is decreased, so that ϵ^2 is much smaller than ϵ, both the left and right sides of the inequality approach $\epsilon|f'(\alpha)|$. That gives $|f(p/q)| \simeq \epsilon|f'(\alpha)|$.

Now factor ϵ out of the inequality to get

$$\left|f\left(\frac{p}{q}\right)\right| \le \epsilon\left(|f'(\alpha)| + \epsilon\frac{|f''(\alpha)|}{2!} + \cdots\right)$$

Suppose $\epsilon < 1$. Then $|f'(\alpha)| + \epsilon\frac{|f''(\alpha)|}{2!} + \cdots$ is less than $|f'(\alpha)| + \frac{|f''(\alpha)|}{2!} + \cdots$. We chose $c = |f'(\alpha)| + \frac{|f''(\alpha)|}{2!} + \cdots$, so then we obtain the desired inequality $\left|f\left(\frac{p}{q}\right)\right| < \epsilon c$.

3d. Use part (b) and (c) to get

$$\frac{1}{q^n} < c\epsilon$$

when $\epsilon < 1$. Substitute for ϵ to get

$$\frac{1}{cq^n} < \left|\alpha - \frac{p}{q}\right|$$

which is the desired inequality.

So, we just need to show that only finitely many rational approximations with $\epsilon \geq 1$ satisfy

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{cq^n}$$

Then

$$1 \leq \frac{1}{cq^n}$$

so $q^n \leq \frac{1}{c}$. That puts a bound on q. However, for each q, there are only finitely many p that satisfy

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{cq^n}$$

since $|\alpha - p/q|$ gets larger as p/q is farther from α. Since there are only finitely many possible q, there are only finitely many p/q that could satisfy this inequality. That proves the desired result.

3e. Let p_i/q_i be the ith convergent of the continued fraction. Then we have $q_0 = 0$, $q_1 = 1$, and $q_i = 2^{i!}q_{i-1} + q_{i-2}$ for $i \geq 2$.

We will prove by induction that $2^{i!} \leq q_i \leq 2^{2(i!)}$ for $i \geq 2$. Checking $i = 2$, we have $q_2 = 2^2$ which does satisfy the inequalities. For the inductive step, first observe that $q_{i+1} = 2^{(i+1)!}q_i + q_{i-1} \geq 2^{(i+1)!}$ as desired. Next, $2^{i!}q_i + q_{i-1} \leq 2^{i!}2^{2(i!)} + 2^{2(i-1)!} \leq 2 \cdot 2^{3(i!)} < 2^{4(i!)}$. For $i \geq 2$, then $2(i+1)! = 2(i+1)(i!) > 4(i!)$, so we obtain $q_{i+1} < 2^{4(i!)} < 2^{2(i+1)!}$, and that completes the induction.

We know that the convergents of a continued fraction alternate being higher and lower than η. Hence

$$\left|\eta - \frac{p_i}{q_i}\right| < \left|\frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}}\right|$$

Using the identity $p_i q_{i+1} - p_{i+1} q_i = \pm 1$,

$$\left|\eta - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}}$$

Use the fact that $q_i \geq 2^{i!}$ to get

$$\left|\eta - \frac{p_i}{q_i}\right| < \frac{1}{2^{i!+(i+1)!}}$$

$$\left|\eta - \frac{p_i}{q_i}\right| < \frac{1}{2^{(i+1)!}}$$

If $c \geq 1$, consider $i + 1 > 2n + \log_2 c$.

$$\left|\eta - \frac{p_i}{q_i}\right| < \left(\frac{1}{2^{i!}}\right)^{i+1}$$

$$< \left(\frac{1}{2^{i!}}\right)^{2n+\log_2 c}$$

$$= \frac{1}{c^{i!}2^{n\cdot 2(i!)}}$$

$$\leq \frac{1}{cq_i^n}$$

which violates (d). This holds for infinitely many i, so it is not possible that η is algebraic. Therefore, η is transcendental.

On the other hand, if $c < 1$, use $i + 1 > 2n$. As before, we obtain

$$\left|\eta - \frac{p_i}{q_i}\right| < \frac{1}{q_i^n}$$

$$< \frac{1}{cq_i^n}$$

which again shows η is transcendental.