HOMEWORK ASSIGNMENT # 6
DUE, Friday, November 14

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. (2 points) Let $n \in \mathbb{Z}$. What is the continued fraction expansion of $\sqrt{n^2+1}$?
2. (a) (3 points) Show that there are infinitely many integers n with the property that $n+1$ and $2n+1$ are perfect squares.
   (b) (2 points) Exhibit two such integers greater than 389.
3. Liouville Numbers. Let α be an algebraic number. That is, a root of a polynomial
$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0,$$
such that $a_i \in \mathbb{Z}$.
   (a) (4 points) Suppose that α is irrational. Show there exists a polynomial $f(x)$ of minimal degree with the following properties:
   - $f(\alpha) = 0$.
   - $f(x)$ has integral coefficients.
   - $f(x)$ has no rational roots.
   - $f'(\alpha) \neq 0$.

   Hint: If you find a polynomial that satisfies the first three conditions but not the last, consider $f'(x)$.
   (b) (2 points) Suppose that $f(x)$ has degree n. Prove that $f\left(\frac{p}{q}\right) \geq \frac{1}{q^n}$ for all rational numbers p/q.
   (c) (4 points) Suppose that
$$\left|\alpha - \frac{p}{q}\right| = \epsilon.$$
Using Taylor's theorem, prove that for small ϵ, $|f(p/q)| \simeq \epsilon |f'(\alpha)|$. In particular, prove that for ϵ sufficiently small, there exists a constant c such that
$$\left|f\left(\frac{p}{q}\right)\right| < c \cdot \epsilon.$$

(d) (1 point) Conclude that all but finitely many rational approximations p/q to α satisfy

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{c \cdot q^n}.$$

(e) (2 points) Prove that the real number

$$\eta = [2^1, 2^2, 2^6, 2^{24}, 2^{120}, \dots, 2^{n!}, \dots] = 2.24902723735498803380399544 9\dots$$

is transcendental. (Hint: using the recurrence relations, estimate p_n and q_n and then use the previous result).