Mathematics 124 Fall 2003

HOMEWORK ASSIGNMENT # 4
DUE, Friday, October 24

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. (3 points) Let p be prime. Prove that

$$\binom{2p}{p} \equiv 2 \mod p^2.$$

2. Let $m \in \mathbb{Z}$.
   (a) (1 point) Let p be any prime divisor of $mx^2 - 1$. Prove that m is a quadratic residue modulo p.
   (b) (4 points) Prove there exists infinitely many primes p such that m is a quadratic residue modulo p. Hint: construct such primes using part (a) above, and recall Euclid's proof that there exist infinitely many prime numbers.
   (c) (3 points) By judicious choices of m above, prove there exist infinitely many primes of the form $1 + 4k$ and $1 + 6k$.

3. Let $p \neq 2, 5$ be prime. Let k be the order of 10 modulo p.
   (a) (2 points) Prove that the decimal expansion of $1/p$ has cycle length exactly k.
   (b) (4 points) Prove that $k < p - 1$ for infinitely many p.
   (c) (3 points) Write the decimal expansion of $1/p$ as

$$1/p = 0.\overline{a_1 a_2 a_3 \dots a_k}.$$

   Suppose that k is even. Prove that

$$a_1 + a_2 + a_3 + \dots a_k = \frac{9k}{2}.$$

   For example, $1/7 = 0.\overline{142857}$ and $1 + 4 + 2 + 8 + 5 + 7 = 54/2 = 27$.
   Hint: notice that $142 + 857 = 999$.