Problem Set 3 Solution Set

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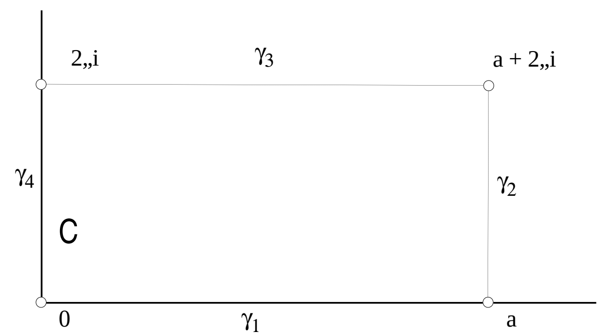
1. Let $f(z) = e^z$. Let a be a positive real number, and let C be the rectangle with vertices $0, a, a + 2\pi i$ and $2\pi i$. Explicitly compute the integral

$$\oint_C f(z)\, dz$$

without using Cauchy's theorem, and verify that Cauchy's theorem applies in this case.



Solution. Colloquially, we write

$$\oint_C = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}.$$

Now we compute the four integrals individually:

$$\int_{\gamma_1} e^z\, dz = \int_0^a e^t\, dt = e^a - 1.$$

$$\int_{\gamma_2} e^z\, dz = \int_0^1 e^{a(1-t)+(a+2\pi i)t} 2\pi i\, dt = e^a e^{2\pi i t}\Big|_0^1 = 0.$$

$$\int_{\gamma_3} e^z\, dz = \int_0^1 e^{(a+2\pi i)(1-t)+2\pi i t}(-a)\, dt = e^a \int_0^1 e^{-at}(-a)\, dt = e^a e^{-at}\Big|_0^1 = 1 - e^a.$$

$$\int_{\gamma_4} e^z\, dz = \int_0^1 e^{2\pi i(1-t)}(-2\pi i)\, dt = e^{-2\pi i t}\Big|_0^1 = 0.$$

Hence

$$\oint_C f(z)\, dz = (e^a - 1) + 0 + 0 + (1 - e^a) = 0.$$

This verifies Cauchy's Integral Theorem since f is analytic inside the region defined by C. □

2. Let γ be the semicircular arc from 1 to -1 in the upper half plane. Use the ML-inequality to prove that

$$\left| \int_\gamma \frac{e^z}{z}\, dz \right| \leq \pi e.$$

Solution. The ML-inequality tells us in this case that

$$\left| \int_\gamma \frac{e^z}{z}\, dz \right| \leq \max_\gamma \left| \frac{e^z}{z} \right| \cdot L(\gamma),$$

where $L(\gamma)$ is the length of γ. We know from Kindergarden that $L(\gamma) = \pi$. Now,

$$\max_\gamma \left| \frac{e^z}{z} \right| = \max_\gamma \frac{|e^z|}{|z|} = \max_\gamma |e^z| = \max_\gamma |e^{x+iy}| = \max_\gamma e^x = e.$$

This gives the desired result. □

3. Let R be the region $\mathbb{C} \setminus \{[0, \infty]\}$. Let $f(z) = \sqrt{z}$, considered as a holomorphic function on R, and such that $f(-1) = i$.

   (a) Let ϵ be a small real number, and let γ_ϵ be the path along the unit circle given explicitly by e^{it} for $t \in [\epsilon, 2\pi - \epsilon]$. Compute the integral

$$I_\epsilon = \int_{\gamma_\epsilon} \sqrt{z}\, dz.$$

   Solution.

$$\begin{aligned} I_\epsilon &= \int_\epsilon^{2\pi-\epsilon} e^{it/2} \cdot ie^{it}\, dt = i \int_\epsilon^{2\pi-\epsilon} e^{3it/2}\, dt \\ &= (2/3)e^{3it/2}\Big|_\epsilon^{2\pi-\epsilon} = (2/3)(e^{3\pi i - 3i\epsilon/2} - e^{3i\epsilon/2}) \\ &= (2/3)(-e^{-3i\epsilon/2} - e^{3i\epsilon/2}) = -\frac{4}{3}\cos\left(\frac{3\epsilon}{2}\right). \end{aligned}$$

□

   (b) Let I be the limit of the integral as ϵ approaches 0. Compute I, and explain why the fact that $I \neq 0$ does not contradict Cauchy's theorem.

   Solution.

$$I = \lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} -\frac{4}{3}\cos\left(\frac{3\epsilon}{2}\right) = -\frac{4}{3}.$$

   The fact that $I \neq 0$ does not contradict Cauchy's theorem because f is multivalued (and hence not analytic) on any simple curve that encloses the origin. Another way to say this is the branch cut we introduced to make f analytic makes the path for I not closed, so that Cauchy's theorem does not apply. □

(c) Compute the real integral

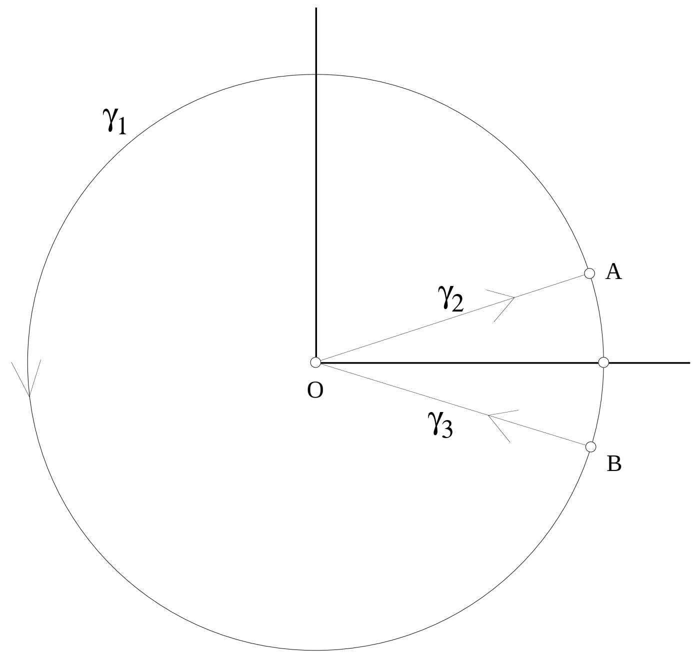
$$J = \int_0^1 \sqrt{x}\, dx.$$

Proof. A Math 1a no-brainer:

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}.$$

□

(d) Prove directly without computing I or J that $I + 2J = 0$.



Solution. Let $A = e^{i\epsilon}$ and $B = e^{i(2\pi-\epsilon)}$ in the figure above. Since $\sqrt{0} = 0$ the function f is analytic in the region area enclosed by $\gamma_1 + \gamma_2 + \gamma_3$. By Cauchy's integral formula,

$$\int_{\gamma_1+\gamma_2+\gamma_3} \sqrt{z}\, dz = 0$$

However,

$$\lim_{\epsilon \to 0} \int_{\gamma_1+\gamma_2+\gamma_3} = \lim_{\epsilon \to 0} \left(\int_{\gamma_2} + \int_{\gamma_2} + \int_{\gamma_3} \right) = \lim_{\epsilon \to 0} \int_{\gamma_1} + \lim_{\epsilon \to 0} \int_{\gamma_2} + \lim_{\epsilon \to 0} \int_{\gamma_3}.$$

Now note that $\lim_{\epsilon \to 0} \int_{\gamma_1} = I$ and $\lim_{\epsilon \to 0} \int_{\gamma_2} = J$. It remains to see what $\lim_{\epsilon \to 0} \int_{\gamma_3}$ evaluates to. Since γ_3 has opposite orientation from γ_2 this integral gains a minus sign with respect to J. However, as $\epsilon \to 0$ the values of $\sqrt{z}$ approach the negative real values of the real square root function. So we pick up a second minus sign. We conlude $\lim_{\epsilon \to 0} \int_{\gamma_3} = J$. Hence $I + 2J = 0$. □

4. If R is a simply connected region with boundary C, prove that

$$A = \frac{1}{2i} \oint_C \overline{z}\, dz,$$

where A is the area of R.

Solution. We apply Green's theorem. Write $\overline{z} = x - iy$, $dz = dx + idy$. All partial derivatives being continuous, we compute

$$\begin{aligned}\oint_C \overline{z}\, dz &= \oint_C x\, dx + y\, dy + i \oint_C -y\, dx + xdy \\ &= \iint_R (0 - 0) dA + i \iint_R (1 - (-1)) dA \\ &= 2iA,\end{aligned}$$

from which the desired equality follows immediately. □