FINAL EXAM

1. Let a_1, $a_2, \ldots$, a_n be complex numbers. Prove there exists a real $x \in [0,1]$ such that
$$\left| 1 - \sum_{k=1}^{n} a_k e^{2\pi i k x} \right| \geq 1.$$

   Let $f(z) = 1 - \sum_{k=1}^{n} a_n z^n$. Then $f(0) = 1$. The Maximum modulus principle says that there exists a z on the circle $|z| = 1$ with $|f(z)| \geq |f(0)| = 1$. For such a z, let $z = e^{2\pi i x}$ with $x \in [0,1]$. Then x satisfies the above inequality.

2. Let $f(z)$ and $g(z)$ be two entire functions satisfying the equation $f^2 + g^2 = 1$. Prove there exists an entire function $h(z)$ such that $f = \cos h$ and $g = \sin h$.

   Suppose that $f = \cos h$ and $g = \sin h$. Then
$$f = \frac{e^{ih} + e^{-ih}}{2}, \qquad g = \frac{e^{ih} - e^{-ih}}{2i}.$$
   Solving for e^{ih}, we find that $e^{ih} = f + ig$. So, for h to exist, we must be able to take the logarithm of $f + ig$, or equivalently (from homework), show that $f + ig$ doesn't vanish. Yet
$$(f + ig)(f - ig) = f^2 + g^2 = 1.$$
   So $f + ig$ can never equal zero. Thus we may take $h = \log(f + ig)/i$, and all the appropriate equations hold.

3. Prove that
$$\int_0^\infty \frac{\log x}{x^4 + 1}\, dx = \frac{-\pi^2}{8\sqrt{2}}$$

   One can use various contours to evaluate this integral. For example, one could take a semicircle in the upper half plane, and let $f(z) = \log(z)/(z^4 + 1)$. The Γ_R and Γ_ϵ integrals vanish, and so one is left with
$$\int_{-\infty}^0 \frac{\log z}{z^4 + 1} + \int_0^\infty \frac{\log z}{z^4 + 1} = 2\pi i \sum \text{residues}.$$
   Or since $\log(-z) = \log(z) + i\pi$,
$$\int_0^\infty \frac{2 \log z}{z^4 + 1} + \int_0^\infty \frac{\pi i}{z^4 + 1} = 2\pi i \sum \text{residues}.$$

The poles in the upper half plane are simple poles at $z_0 := e^{\pi i/4}$ and $z_1 := e^{3\pi i/4}$. The residue at z_0 is

$$\lim_{z\to z_0} \frac{\log z(z-z_0)}{z^4+1} = \lim_{z\to z_0} \frac{\log z}{4z^3} = \frac{(\pi i/4)}{4} \cdot e^{-3\pi i/4} = \frac{\pi}{16\sqrt{2}}(1-i).$$

The residue at z_1 is

$$\lim_{z\to z_1} \frac{\log z(z-z_1)}{z^4+1} = \lim_{z\to z_1} \frac{\log z}{4z^3} = \frac{(3\pi i/4)}{4} \cdot e^{-9\pi i/4} = \frac{3\pi}{16\sqrt{2}}(1+i).$$

Thus

$$\int_0^\infty \frac{2\log z}{z^4+1} + \int_0^\infty \frac{\pi i}{z^4+1} = 2\pi i \cdot \frac{\pi}{16\sqrt{2}}(1-i+3(1+i)) = \frac{(4i-2)\pi^2}{8\sqrt{2}}.$$

Equating real and imaginary parts we find that

$$\int_0^\infty \frac{\log z}{z^4+1} = \frac{-\pi^2}{8\sqrt{2}}, \qquad \int_0^\infty \frac{1}{z^4+1} = \frac{\pi}{2\sqrt{2}}.$$

4. Let $a \geq b \geq 0$. Evaluate the integral

$$\int_{-\infty}^\infty \frac{\sin ax \sin bx}{x^2}\, dx$$

An elementary calculation shows that

$$\sin ax \sin bx = \text{Real Part}\left\{\frac{e^{i(a+b)x} - e^{i(a-b)x}}{-2}\right\}.$$

We calculate the Cauchy Principle Value of the integral

$$\int_{-\infty}^\infty \frac{e^{i(a+b)x} - e^{i(a-b)x}}{-2x^2}\, dx.$$

Consider the usual upper half plane contour. Note that $a \geq b \geq 0$ so the exponents are *positive*, thus we may apply Jordan's lemma (if $a = b$, the second term is 1, so Jordan's Lemma doesn't apply; however we instead use the ML inequality applied to $1/x^2$ to show that the Γ_R integral vanishes in this case). Note also that the function has a SIMPLE pole at 0 so the Cauchy Principle Value exists; Many people worked with expressions like $\int_{-\infty}^\infty e^{imx}/x^2 dx$; such expressions *do not have* a Cauchy Principle Value. The only contribution to the integral comes from the simple pole at zero. Since *it is a simple pole*, we can evaluate the half circle loop around zero by taking half the residue. We find that

$$\int_{-\infty}^\infty \frac{\sin ax \sin bx}{x^2}\, dx = \text{Real}\left\{\pi i \cdot \left(\frac{i(a+b)}{-2} - \frac{i(a-b)}{-2}\right)\right\} = \pi b.$$

5. Let $f(x+iy) = u(x,y) + iv(x,y)$ be an entire function. Suppose that $u(x,y)$ is negative for all $x+iy \in \mathbb{C}$. Prove that f is constant.

One has $|e^f| = |e^{u+iv}| = e^u < 1$. So by Liouville's theorem, e^f and thus f constant. One could also use the inequality $|1/(1-f)| \leq 1$. Trying to prove that $1/f$ was bounded was a bad choice.

6. Let n be a positive integer, and let $\omega = e^{2\pi i/n}$. Let $f(z)$ be an entire function such that $f(z) = f(\omega z)$. Prove there exists an entire function $h(z)$ such that $f(z) = h(z^n)$.

Consider the Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of $f(z)$. Since $f(\omega z) = f(z)$, comparing Taylor series expansions we find that $\omega^k a_k = a_k$, and thus $a_k = 0$ unless $n|k$. In particular, If

$$h(z) = a_0 + a_n z + a_{2n} z^n + a_{3n} z^3 + \dots$$

Then $h(z^n) = f(z)$. Since $f(z)$ is entire, the Taylor series converges for all z, so a simple check shows that $h(z)$ is entire also.

7. Let $0 < \alpha, \beta < 1$. Evaluate the integral

$$\int_0^{\infty} \frac{x^{-\alpha} - x^{-\beta}}{x-1} \, dx$$

Hint: First consider the Cauchy principle value of the integral

$$\int_0^{\infty} \frac{x^{-\alpha}}{x-1} \, dx$$

Let $f(z) = e^{-\alpha \log(z)}/(z-1)$. Using the key hole contour with a bump at $z = 1$, we find that

$$0 = \text{cpv.} \int_{0^+}^{\infty^+} + \text{cpv.} \int_{\infty^-}^{0^-} + \lim_{\epsilon \to 0} \left(\int_{1^+ - \epsilon}^{1^+ + \epsilon} + \int_{1^- + \epsilon}^{1^- - \epsilon} \right).$$

Here $a^{\pm}$ refers to a on the real line either side of the branch cut. We find then that if I is our integral, that

$$I(1 - e^{-2\pi i \alpha}) = \left(\int_{1^+ + \epsilon}^{1^+ - \epsilon} + \int_{1^- - \epsilon}^{1^- + \epsilon} \right).$$

Since these are half integrals through *simple* poles, we may evaluate them by taking half the residue. Thus

$$I(1 - e^{-2\pi i \alpha}) = \pi i (1 + e^{-2\pi i \alpha})$$

and so

$$I = \pi \cdot i \cdot \frac{1 + e^{-2\pi i \alpha}}{1 - e^{-2\pi i \alpha}} = \pi \cot \pi\alpha.$$

Thus

$$\int_0^\infty \frac{x^{-\alpha} - x^{-\beta}}{x - 1}\, dx = \pi(\cot \pi\alpha - \cot \pi\beta).$$