Mathematics 113 Fall 2002

FINAL EXAM
Three Hours

Requirements: Calculators are not allowed. Show all your working, and submit partial solutions. Write as neatly as possible. Partial marks will be awarded, when appropriate. If you use a theorem, make sure to state it. All questions are worth the same amount. Do not feel you need to write up the questions in order.

1. Let $a_1, a_2, \ldots, a_n$ be complex numbers. Prove there exists a real $x \in [0,1]$ such that
$$\left|1 - \sum_{k=1}^{n} a_k e^{2\pi i k x}\right| \geq 1.$$

2. Let $f(z)$ and $g(z)$ be two entire functions satisfying the equation $f^2 + g^2 = 1$. Prove there exists an entire function $h(z)$ such that $f = \cos h$ and $g = \sin h$.

3. Prove that
$$\int_0^\infty \frac{\log x}{x^4+1}\,dx = \frac{-\pi^2}{8\sqrt{2}}$$

4. Let $a \geq b \geq 0$. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\sin ax \sin bx}{x^2}\,dx$$

5. Let $f(x+iy) = u(x,y) + iv(x,y)$ be an entire function. Suppose that $u(x,y)$ is negative for all $x+iy \in \mathbb{C}$. Prove that f is constant.

6. Let n be a positive integer, and let $\omega = e^{2\pi i/n}$. Let $f(z)$ be an entire function such that $f(z) = f(\omega z)$. Prove there exists an entire function $h(z)$ such that $f(z) = h(z^n)$.

7. Let $0 < \alpha, \beta < 1$. Evaluate the integral
$$\int_0^\infty \frac{x^{-\alpha} - x^{-\beta}}{x-1}\,dx$$
Hint: First consider the Cauchy principle value of the integral
$$\int_0^\infty \frac{x^{-\alpha}}{x-1}\,dx$$