EXTRA PROBLEMS

1. Let n be a positive integer. Evaluate the integral

$$\int_0^{2\pi} \frac{\sin n\theta}{\sin\theta}\, d\theta.$$

2. Let $\alpha > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{1+x+x^2}\, dx.$$

3. Does there exist a function $f(z)$ holomorphic for all $z \in \mathbb{C} \setminus \{0\}$ which satisfies the following inequality?

$$|f(z)| \geq \frac{1}{\sqrt{|z|}}.$$

4. Let $f(z)$ and $g(z)$ be holomorphic functions such that $f^2 = g^3$. Prove there exists a holomorphic function $h(z)$ such that $f = h^3$ and $g = h^2$.

5. Let $f(z)$ be an entire function with no zeros or poles such that for $|z| = R$ sufficiently large, $|f(z)| < R^R$. Prove that $f = Ae^{Bz}$ for some constants A and B.

6. Let R be a closed region inside $\mathbb{C}$. Prove that the minimum absolute value of e^z on R is obtained on the boundary of R.

7. Let α be a real number satisfying $\alpha > 1$. Compute the integral

$$\int_0^{\infty} \frac{1}{1+x^{\alpha}}\, dx.$$

8. Prove that

$$\int_0^{\infty} \frac{(\log x)^2}{x^2+1}\, dx = \frac{\pi^3}{8}.$$

9. Consider the function $e^{z(t+1/t)/2}$ as a holomorphic function in the variable t, and let

$$e^{z(t+1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

be its Laurent expansion in the Annulus $1 < t < 2$.

(a) Prove that the sum converges for all $0 < z < \infty$.

(b) Prove that $J_n(z)$ is a holomorphic function as a function of z, and find $J_n(0)$.

10. Let m be a positive integer.

    (a) Prove that
    $$\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\cdots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}.$$

    (b) Prove that
    $$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(\frac{m-1}{m}\right) = \sqrt{\frac{(2\pi)^{m-1}}{m}}$$

11. Let $f(z)$ be a meromorphic function with finitely many poles. Suppose that
    $$\lim_{z\to\infty} zf(z) = 1.$$
    Find the sum of the residues of f at all its poles.

12. Which of the following conditions determine an entire holomorphic function uniquely? Give a brief explanation for a positive response, and a counter example otherwise.

    (a) The value of f at n for every integer n.

    (b) The value of f at $1/n$ for every integer n.

    (c) The poles of f, and the residue at each pole.

    (d) The value of f and all its derivatives at 0.

13. Prove that the series
    $$z^{1!} + z^{2!} + z^{3!} + \ldots$$
    has the natural boundary $|z| = 1$.

14. Let $f^{(n)}$ denote the nth derivative of f. Does there exist a function f holomorphic in a neighbourhood of zero such that $f^{(n)}(0) = n^n$ for all positive integers n?

15. Let $f(z)$ be an entire function such that
    $$\lim_{z\to\infty} \frac{f(z)}{z^n} = 0.$$
    Prove that $f(z)$ is a polynomial of degree $\leq n-1$.