# math123, Abstract Algebra II

## PROBLEM SET 6

- Exercise 9.5.8 from Artin's book
- Exercise 9.5.9
- Exercise 9.5.13
- Exercise 9.6.8
- Exercise 9.8.3
- Solve the following

  **Problem 1**:

  Let $\rho$ be a $d$–dimensional representation of a finite group $G$, and let $\sigma$ be a ne–dimensional representation of $G$. Define $\rho'_g = \sigma_g \rho_g$, where the right hand side stands for the product of a scalar and a matrix. Show that $\rho'$ is a representation and describe its character.

- The following problem is rather long, but (I hope) I am giving you enough hints to make it almost straightforward.

  **Problem 2**:

  We want to prove that, if $V$ is a finite dimensional irreducible representation of a finite group $G$, then the dimension of $V$ divides the order of the group $G$:

  $$\dim V \ \mid \ |G| \ .$$

  *Notation and review:*

  From now on, we will fix a group of order $N$, $G = \{g_1, g_2, \dots, g_N\}$. We will denote by $C_1, \dots, C_s$ its $s$ conjugacy classes:

  $$G = C_1 \cup C_2 \cup \cdots \cup C_s$$

  For every conjugacy class $C_i \subset G$, we define the element

  $$\gamma_i \ = \ \sum_{g \in C_i} g \ .$$

  We know then that there are $s$ (up to isomorphism) irreducible representations, $V_1, \dots, V_s$, of dimensions $d_1, \dots, d_s$ respectively. We thus want to prove that $d_i \mid N \ , \quad \forall i = 1, \dots, s$. Recall the definition of the *regular representation*:

  $$V^{reg} \ = \ \{\sum_{i=1}^{N} c_i g_i \ ; \quad c_i \in \mathbb{C}\} \ ,$$

  with action of $G$ by left multiplication. We know that $V^{reg}$ admits the following decomposition as direct sum of irreducible subrepresentations:

  $$V^{reg} \ = \ V_1^{\oplus d_1} \oplus \cdots V_s^{\oplus d_s} \ .$$

  We will denote $R_i = V_i^{\oplus d_i}$, so that

  $$V^{reg} = R_1 \oplus \cdots \oplus R_s \ .$$

Notice that the vector space $V^{reg}$ has naturally the structure of a ring (infact, of an algebra over $\mathbb{C}$), with multiplication law:

$$\sum_{i=1}^{N} c_i g_i \cdot \sum_{j=1}^{N} d_j g_j \;=\; \sum_{i,j=1}^{N} c_i d_j (g_i g_j) \ .$$

This algebra is usually called the *group algebra* of $G$ and it is denoted by $\mathbb{C}[G]$. We will denote by $Z(G)$, the *center* of $\mathbb{C}[G]$, namely the collection of elements of $V^{reg}$ which commute with everything:

$$Z[G] \;=\; \{v \in V^{reg} \mid av = va \ , \ \ \forall a \in V^{reg}\} \ .$$

(Equivalently, the elements in $Z[G]$ are preserved by the adjoint action of $G$: $gvg^{-1} = v \ , \ \forall g \in G$).

(1) In (1)–(2) we want to find a basis for $Z[G] \subset V^{reg}$. Prove that $\gamma_i \in Z[G] \ , \ \forall i = 1, \dots, s$.

(2) Prove that, in fact, $\{\gamma_i \ , \ i = 1, \dots, s\}$ form a basis of $Z[G]$.

(3) In (3)–(9) we want to study the ring structure of $V^{reg}$. Prove that $R_i \;=\; V_i^{\oplus d_i} \subset V^{reg}$ is a left ideal:

$$aR_i \subset R_i \ , \ \forall a \in V^{reg} \ .$$

(4) Prove that $R_i$ is invariant under the adjoint action of $G$:

$$gR_i g^{-1} \subset R_i \ .$$

(5) Conclude that $R_i$ is both a left and right ideal:

$$aR_i \subset R_i \ , \ \ R_i a \subset R_i \ , \ \ \forall a \in V^{reg} \ .$$

In particular $ab = 0$ if $a \in R_i$ and $b \in R_j$ with $i \neq j$.

(6) Prove that there is a ring homomorphism $\phi_i : \ R_i \to \mathrm{End}_{\mathbb{C}}(V_i)$, given by the multiplication law: $R_i \times V_i \to V_i$.

(7) Prove that $\phi_i$ is injective. (**Hint:** notice that, if $a \in \ker \phi_i$, then $av = 0$, $\forall v \in V^{reg}$; deduce that $a = 0$).

(8) By counting dimensions, show that $\phi$ is also surjective. In conclusion $R_i \simeq \mathrm{End}_{\mathbb{C}}(V_i)$ as a $\mathbb{C}$–algebra.

(9) Let us denote by $e_i \in R_i$ the element corresponding to the identity operator $\mathbb{1} \in \mathrm{End}_{\mathbb{C}}(V_i)$ (under the isomorphism $\phi_i$). Show that, the map $V^{reg} \ \to V^{reg}$ given by

$$v \mapsto e_i v$$

is the natural projection $V^{reg} \twoheadrightarrow R_i$. (In particular, $e_i^2 = e_i$).

(10) Prove that $e_i \in Z[G]$. In conclusion, we have found a new basis $\{e_1, \dots, e_s\}$ of $Z[G]$.

(11) In (11)–(13) we want to study the relation between the two bases $\{\gamma_1, \dots, \gamma_s\}$ and $\{e_1, \dots, e_s\}$ of $Z[G]$. Namely we want to express each $e_i$ as linear combination of $\gamma_i$'s. Prove that every $v \in V^{reg}$ can be decomposed as:

$$v = \frac{1}{N} \sum_{g \in G} \chi^{reg}(ag^{-1}) g \ .$$

(12) Prove the following equation:

$$\chi^{reg}(e_i g^{-1}) \;=\; d_i \chi_i(g^{-1}) \ .$$

(13) In conclusion, we have:

$$e_i \;=\; \frac{d_i}{N}\sum_{g\in G}\chi_i(g^{-1})g \;.$$

This is the decomposition we wanted.

(14) Now we are ready to conclude the proof of our theorem. Prove that:

$$e_i \;=\; \frac{d_i}{N}\sum_{g\in G}\chi_i(g^{-1})ge_i \;.$$

(15) Prove that every character $\chi_i(g)$ is linear combination of $N$–th roots of unity with integer coefficients (Recall, we more or less proved it in class); namely

$$\chi_i(g) \;\in\; \mathrm{span}_{\mathbb{Z}}\{1,\zeta_N,\dots,\zeta_N^{N-1}\} \;.$$

(16) Deduce that

$$\frac{N}{d_i}e_i \;\in\; \mathrm{span}_{\mathbb{Z}}\{\zeta_N^k ge_i \;,\; k=1,\dots,N-1 \;,\; g\in G\} \;=\; M \;.$$

(17) Deduce that multiplication by $\frac{N}{d_i}$ leaves $M$ invariant.

(18) Conclude that $\frac{N}{d_i}\in\mathbb{Z}$, as we wanted.