math123, Abstract Algebra II

PROBLEM SET 5

- Exercise 9.1.1 from Artin's book
- Exercise 9.1.4
- Exercise 9.1.6
- Exercise 9.2.2
- Exercise 9.2.3
- Exercise 9.2.7
- Exercise 9.3.4
- Exercise 9.4.4
- Solve the following

  Problem 1:

  We want to study the standard 3–dimensional representation of the group T of isometries of the regular tetrahedron. (**Note:** this problem is more or less solved in the book, but we want to do carefully all the computations).

  (1) Convince yourself that the following vectors:
  $$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ , \ v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \ , \ v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \ , \ v_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \ ,$$
  are the four vertices of a regular tetrahedron in $\mathbb{R}^3$. Draw the corresponding tetrahedron.

  (2) Write all 12 elements of T as 3×3 orthogonal matrices.

  (3) Prove that these matrices define a faithful, irreducible, 3–dimensional complex representation of T.

  (4) Find all conjugacy classes of T, and write the class equation.

  (5) Compute the character χ of this representation, and show that it is a class function.
- Solve the following

  Problem 2:

  As in the previous problem, we want to study the standard 2–dimensional representation of the groups C_n and D_n of symmetries of the regular n–gon on the plane.

  (1) Recall the definition of the groups C_n and D_n in terms of generators and relations, and list all the elements.

  (2) Find all conjugacy classes of C_n and D_n.

  (3) Consider the standard 2–dimensional representation of C_n:
  $$\rho^c \ : \ C_n \ \to SO_2 \subset GL_2 \ .$$
  and of D_n:
  $$\rho^d \ : \ D_n \ \to O_2 \subset GL_2 \ .$$
  Write down the 2×2 matrices corresponding to all the elements of C_n and D_n.

(4) a) Prove that ρ^c and ρ^d are irreducible representations of C_n and D_n respectively, if considered as real representations.
b) Notice that C_n is abelian, hence the complex representation ρ^c is necessarily reducible. Find explicitely the decomposition of $\mathbb{C}^2$ as direct sum of two 1–dimensional irreducible subrepresentations of C_n.
c) Find for which values of n the complex 2–dimensional representation ρ^d of D_n is irreducible.

(5) a) Compute the character χ of ρ^c, and verify that it is the sum of the characters of the two irreducible subrepresentations of $\mathbb{C}^2$.
b) Compute the character χ of ρ^d, and verify that it is a class function.