math123, Abstract Algebra II

PROBLEM SET 4

- Exercise 8.5.3 from Artin's book
- Exercise 8.5.4
- Exercise 8.5.10
- Exercise 8.6.11 (Note: there is a typo in the formula)
- Exercise 8.6.13
- Solve the following
  Problem 1:
  Consider a generic matrix
  $$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$$
  Write it explicitely as a product $P = AB$, with $A \in SO_2(\mathbb{R})$ and $B \in H$, and check that the matrix elements of A and B are continuous functions of a, b, c, d. (**Note:** this shows that the map $f^{-1} : SL_2(\mathbb{R}) \to SO_2(\mathbb{R}) \times H$ is a continuous map (As we claimed in class).
- Solve the following
  Problem 2:
  Recall that if matrices A and B commute, then $e^A e^B = e^{A+B}$. Consider the following 2×2 matrices:
  $$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$
  Compute: $e^A e^B$, $e^B e^A$, e^{A+B}. Explain why this result agrees with the above statement.
- Solve the following
  Problem 3:
  The goal of this exercise is to show how to compute the exponential of any matrix $A \in Mat_{n\times n}(\mathbb{C})$.
  (1) Compute e^A for a diagonal matrix
  $$A = \begin{bmatrix} \lambda_1 & & & \mathrm{O} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathrm{O} & & & \lambda_n \end{bmatrix}$$
  (2) More in general, write the exponential of a block diagonal matrix
  $$A = \begin{bmatrix} A_1 & & & \mathrm{O} \\ & A_2 & & \\ & & \ddots & \\ \mathrm{O} & & & A_k \end{bmatrix} ,$$
  with $A_i \in Mat_{n_i\times n_i}(\mathbb{C})$, $n_1 + \cdots + n_k = n$, in terms of the exponential of the matrices $A_1, \ldots A_k$.

(3) Compute the exponential of the following $n \times n$ matrix:

$$N_n = \begin{bmatrix} 0 & 1 & & & O \\ & 0 & 1 & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ O & & & & 0 \end{bmatrix}$$

(1's only on the diagonal next to the principal diagonal).

(4) More in general, compute the exponential of the following $n \times n$ matrix:

$$J_n(\lambda) = \lambda \mathbb{1}_n + N_n = \begin{bmatrix} \lambda & 1 & & & O \\ & \lambda & 1 & & \\ & & \ddots & 1 & \\ & & & \lambda & 1 \\ O & & & & \lambda \end{bmatrix}$$

(**Hint:** the identity matrix commutes with any matrix!)

(5) Let $A \in Mat_{n\times n}(\mathbb{C})$, $P \in GL_n(\mathbb{C})$ and $B = PAP^{-1}$. Prove the following identity:

$$e^B = Pe^A P^{-1}.$$

(6) Recall that any matrix $A \in Mat_{n\times n}(\mathbb{C})$ can be reduced, after a change of basis, to the *Jordan canonical form*; namely there is $P \in GL_n(\mathbb{C})$ such that

$$A = P \begin{bmatrix} J_{n_1}(\lambda_1) & & & O \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ O & & & J_{n_k}(\lambda_k) \end{bmatrix} P^{-1} .$$

Use this fact and the above results to compute, for an arbitrary complex $n \times n$ matrix A, the exponential e^A.

(7) As a corollary, prove the following identity:

$$\det e^A = e^{\mathrm{Tr} A}$$

- Solve the following

Problem 4

Prove that if $P \in SL_n(\mathbb{R})$, then $\det P = +1$.

Note: You can solve this problem whichever way you like. For example by solving Exercise 8.1.13 in the book. My suggestion is to follow the following:

Hints:

(1) Show that $\det P = \pm 1$.

(2) From now on, until the last step, we will assume that P is diagonalizable over $\mathbb{C}$. Prove that, if $\lambda \in \mathbb{C}$ is eigenvalue with multiplicity k, then $\bar{\lambda}$ is also eigenvalue with same multiplicity k. Conclude that, if P has no real eigenvalue, then $\det P > 0$ (and thus $\det P = +1$).

(3) Suppose now $\lambda \in \mathbb{R}$ is a real eigenvalue of P, with eigenvector v.

- Show that the space

$$U = \{u \in \mathbb{R}^n \mid < u, v >= 0\} = v^\perp ,$$

is an $(n-1)$-dimensional space left invariant by P (Here $< u, v >$ denotes the skew-symmetric bilinear form which defines SP_n). Namely $PU \subset U$.

- Assuming that P is diagonalizable, prove that there is a complementary 1-dimensional invariant subspace $U' = \mathbb{R}v'$.
- Prove that v' is eigenvector of P with eigenvalue λ^{-1}.
- Prove that the $(n-2)$-dimensional subspace orthogonal to v and v' is left invariant by P.
- Use induction to show that all real eigenvalues (with multiplicities) come in pairs λ, λ^{-1}.
- Deduce that $\det P = +1$.

(4) Finally, if $P \in SP_n(\mathbb{R})$ is not diagonalizable, argue that one can "shift it a little" to get some other diagonalizable $P' \in SP_n(\mathbb{R})$. Deduce that $\det P = 1$.