

math123, Abstract Algebra II  
Final Exam

Your name:

Problem	Points	Your Grade
1	10	
2	10	
3	15	
4	15	
5	15	
6	10	
7	15	
8	10	
9	10	
10	10	
11	15	
12	15	
<b>Total</b>	150	

**Problem 1** (pt 10)

Given  $a, b, c, d \in \mathbb{R}$ , consider the map  $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$  defined by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \mapsto a\alpha_1\beta_1 + b\alpha_2\beta_2 + c\alpha_1\beta_2 + d\alpha_2\beta_1$$

For which values of  $a, b, c, d \in \mathbb{R}$  this map defines a:

- (a) bilinear form on  $\mathbb{R}^2$ ?
- (b) symmetric bilinear form on  $\mathbb{R}^2$ ?
- (c) positive definite (symmetric) bilinear form on  $\mathbb{R}^2$ ?

**Problem 2** (pt 10)

Let  $V$  be a vector space over  $\mathbb{R}$  with a positive definite, symmetric, bilinear form  $(\cdot, \cdot)$ . Suppose  $\{v_1, \dots, v_n\} \subset V$  is an orthonormal system, namely a collection of vectors (not necessarily spanning  $V$ ) such that:

$$(v_i, v_j) = \delta_{i,j} .$$

(a) Prove that

$$\sum_{i=1}^n (v_i, v)^2 \leq (v, v) , \quad \forall v \in V .$$

(b) Prove that  $\sum_{i=1}^n (v_i, v)^2 = (v, v)$ ,  $\forall v \in V$  if and only if  $\{v_1, \dots, v_n\}$  is a basis of  $V$ .

**Problem 3** (pt 15)

- (a) Describe the Lie algebra  $sl_2(\mathbb{C})$  of the Lie group  $SL_2(\mathbb{C})$  (No proof is required in this part of the problem).
- (b) Consider the adjoint representation of  $SL_2(\mathbb{C})$  on its Lie algebra  $sl_2(\mathbb{C})$ . Find a non-degenerate, symmetric, bilinear form on  $sl_2(\mathbb{C})$ , which is invariant under the action of  $SL_2(\mathbb{C})$ .
- (c) Deduce that there is a group homomorphism

$$SL_2(\mathbb{C}) \rightarrow O_3(\mathbb{C}) .$$

**Extra Credit** (pt 5)

- (d) Prove that the kernel of this homomorphism is:  $\{\pm \mathbb{1}\} \subset SL_2(\mathbb{C})$ .
- (e) Argue that the image of this homomorphism has to be in  $SO_3(\mathbb{C})$ .
- (f) Accept (without proving it) that the image is equal to  $SO_3(\mathbb{C})$ , and deduce that there is a group isomorphism:

$$SL_2(\mathbb{C})/\{\pm \mathbb{1}\} \xrightarrow{\sim} SO_3(\mathbb{C}) .$$

**Problem 4** (pt 15)

- (a) Determine the character table of the symmetric group  $G = S_3$ .

**Extra Credit** (pt 5)

- (b) The group  $G$  acts on itself by conjugation. Determine the character  $\chi$  of the associated representation  $\rho$ . (**Note:** it is not the regular representation).
- (c) Write  $\chi$  as a sum of irreducible characters.
- (d) Write the representation  $\rho$  as a direct sum of the irreducible representations of  $G$ .

**Problem 5** (pt 15)

1. Let  $G$  be a group with a finite-dimensional representation  $\rho$  on a vector space  $V$ .
  - (a) Define a ring structure on the space  $E(V)$  of all  $G$ -invariant linear transformations  $T : V \rightarrow V$ .
  - (b) What can you say about  $E(V)$  if  $V$  is an irreducible representation?
2. Let  $R$  be a ring and let  $M$  be a module over  $R$ .
  - (a) Define a ring structure on the space  $E(M)$  of all module homomorphisms  $T : M \rightarrow M$ .
  - (b) Prove that, if  $M$  is irreducible, then every non-zero element  $T \in E(M)$  is invertible.

**Problem 6** (pt 10)

Give a classification of all abelian groups  $G$  of order 1000, with the condition that  $50g = 0, \forall g \in G$  (I am using additive notation for  $G$ ).

**Problem 7** (pt 15)

Let  $V$  be a module over the ring  $R = \mathbb{Q}[t]$ , with generators  $v_1, v_2$  and relations:

$$\begin{aligned}v_1 + tv_2 &= 0, \\t^2(1 - t^3)v_1 + (1 - t^3)v_2 &= 0.\end{aligned}$$

- (a) Find  $v_0 \in V$  such that  $V$  is generated by  $v_0$  (Express it in terms of  $v_1$  and  $v_2$ ).
- (b) Find a generator of the ideal  $I \subset \mathbb{Q}[t]$  consisting of elements  $g \in \mathbb{Q}[t]$  such that  $gv_0 = 0$ .
- (c) What is the dimension of  $V$  as vector space over  $\mathbb{Q}$ ?
- (d) Choose a basis for  $V$  over  $\mathbb{Q}$  and write the matrix for the linear transformation  $T(v) = tv$ .

**Problem 8** (pt 10)

Can  $\sqrt[8]{7}$  be constructed by ruler and compass? (Explain your answer)

**Problem 9** (pt 10)

Let  $P(x) \in \mathbb{Q}[x]$  be a polynomial with a root  $\alpha = a + \sqrt{mb}$ , where  $m \in \mathbb{Z}$  is square-free and  $a, b \in \mathbb{Q}$ . Prove or disprove:  $\beta = a - \sqrt{mb}$  is a root of  $P(x)$ .

**Problem 10** (pt 10)

Determine the number of monic irreducible polynomials of degree 3 over  $\mathbb{F}_5$ .

**Problem 11** (pt 15)

Let  $\mathbb{Q} \subset K$  be a splitting field extension for the polynomial  $x^n - 1$ .

- (a) Find an isomorphism between the Galois group  $G(K/\mathbb{Q})$  and the cyclic group of order  $\Phi(n)$  (Where  $\Phi(n)$  denotes the Euler's function, namely the number of integers  $k = 1, \dots, n$  which are relatively prime to  $n$ ).
- (b) Prove that  $[K : \mathbb{Q}] = \Phi(n)$ .
- (c) Prove that the polynomial  $x^n - 1$  admits the following irreducible factor of degree  $\Phi(n)$ :

$$P(x) = \prod_{\gcd(k,n)=1} (x - e^{2k\pi i/n}) \in \mathbb{Q}[x].$$

**Problem 12** (pt 15)

Consider the polynomial  $P(x) = x^5 - 2 \in \mathbb{Q}[x]$ .

- (a) What are the five complex roots of  $P(x)$ ?
- (b) Let  $K$  be the splitting field for  $P(x)$  over  $\mathbb{Q}$ . Determine the degree of  $K$  over  $\mathbb{Q}$ .
- (c) Suppose  $g(x) \in \mathbb{Q}[x]$  is a polynomial irreducible of degree 4, with all its roots in  $K$ . What is the degree of the splitting field for  $g(x)$  over  $\mathbb{Q}$ ?

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