math123, Abstract Algebra II

Exam 2

Your name:

Problem 1 (15pt)

Let H be a normal subgroup of index 2 of a finite group G, and let $\rho:\ G \to GL(V)$ be a finite–dimensional representation of G. Define ρ' by:

$$\begin{aligned} \rho'(h) &= \rho(h)\ , \quad \text{for } h \in H\ , \\ \rho'(g) &= -\rho(g)\ , \quad \text{for } h \notin H\ . \end{aligned}$$

(1) Show that ρ' is a representation of G.
(2) Show that ρ' is irreducible if and only if ρ is irreducible.

Problem 2 (20pt)

Let ρ be an irreducible representation of a finite group G on a vector space V, and let T be the linear operator on V defined by

$$T \;=\; \sum_{g\in G} \rho_g \;.$$

(1) Prove that T is an invariant operator on V.
(2) What does Schur's Lemma tell us about T?
(3) Using the orthogonality relations, show that if ρ is the non–trivial representation, then the trace of T is zero.
(4) Prove that, is ρ is not the trivial representation, then $T = 0$.

Problem 3 (20pt)

(1) Determine the character table for the Klein Four Group $K = C_2 \times C_2$.
(2) Consider a group G of class equation $8 = 1 + 1 + 2 + 2 + 2$. Determine its character table. (**Hint.** What is the center of the group? And the quotient by the center?)
(3) Let V be a representation of G with character:

	(1)	(1)	(2)	(2)	(2)
χ	8	0	4	2	2

Write the decomposition of V as direct sum of irreducible representations of G.

Problem 4 (15pt)

Let R be a Euclidean domain, and let M be a finitely generated module over R. For each of the following statements, determine whether it is necessarily true (by proving it) or it can be false (by finding a counterexample).

(1) M has a basis.
(2) M is a submodule of a free module R^n.
(3) M is a quotient module of a free module R^n.
(4) If M is free of rank n, and we have a submodule $R^k \subset M$, then M/R^k is free of rank $n-k$.
(5) If M is free of rank n, and we have a bijective module homomorphism $R^k \to M$, then $k = n$.

Problem 5 (15pt)

Let $L \subset \mathbb{Z}^2$ be the sublattice spanned by the vectors

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad , \quad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} .$$

Find bases $\mathcal{B} = [e_1, e_2]$ of $\mathbb{Z}^2$ and $\mathcal{B}' = [v_1, v_2]$ of L, so that $v_1 = d_1 e_1$ and $v_2 = d_2 e_2$, with $d_1, d_2 \in \mathbb{Z}_+$.

Problem 6 (15pt)

Let V be the abelian group generated by three elements x, y, z, with relations

$$\begin{aligned} 6x + 4y + 4z &= 0 \ , \\ 2x + 2y + 8z &= 0 \ . \end{aligned}$$

Identify V as a direct sum of cyclic groups.