math123, Abstract Algebra II

Exam 1

Your name:

Problem 1 (15pt)

Let V be a real vector space, with a given basis $\mathcal{B} = [v_1, v_2, \ldots, v_n]$. Recall that to every vector $v \in V$, you can associate the coordinate vector $X \in \mathbb{R}^n$, defined by $v = \mathcal{B}X$. Let $< \, , \, >$: $V \times V \longrightarrow \mathbb{R}$ be a bilinear form on V.

(1) Show how to write the matrix A of $< \, , \, >$ with respect to the basis $\mathcal{B}$.
(2) Consider a change of basis $\mathcal{B}' = \mathcal{B}P$. Write the matrix A' of $< \, , \, >$ in the new basis $\mathcal{B}'$ in terms of A and P.
(3) (**extra credit**, +5pt) Prove that the change of basis formula you gave for (2) is correct.

Problem 2 (18pt)

Let < , > be a bilinear form on a real vector space V of dimension 3. Suppose in a given basis $\mathcal{B} = [v_1, v_2, v_3]$ of V, the matrix A of the bilinear form < , > is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let $\mathcal{B}'$ be a new basis of V, and let A' be the matrix of < , > in basis $\mathcal{B}'$. Which of the following statements are necessarily true and which are not (you do NOT need to prove or justify your answers):

(1) A' has eigenvalues $1, 2, -1$.
(2) $\exists P \in O_3(\mathbb{R})$ such that $PA'P^T$ is diagonal.
(3) A' has one negative eigenvalue.

Problem 3 (15pt)

Let $A,\ B$ be positive definite hermitian matrices. Determine which of the following matrices are positive definite hermitian matrices (you do NOT need to prove or justify your answers):

(1) A^2,
(2) A^{-1},
(3) AB,
(4) $A+B$.

Problem 4 (17pt)

1. State Sylvester's law for real symmetric $n \times n$ matrices.
2. Rephrase it in terms of the orbits of the action of the group $GL_n(\mathbb{R})$ on the space S of all real symmetric $n \times n$ matrices given by:

$$\begin{array}{ccc} GL_n(\mathbb{R}) \times S & \longrightarrow & S \\ P\,,\ A & \mapsto & PAP^T \end{array}$$

Problem 5 (15pt)

Let A be any real symmetric matrix. Prove that e^A is real, symmetric and positive definite.

Problem 6 (15pt)

Let G be the linear group

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, \quad x > 0, y \in \mathbb{R} \right\} \subset GL_2(\mathbb{R})$$

(1) Determine all the 2×2 real matrices A such that $e^{tA} \in G$, $\forall t \in \mathbb{R}$ (You do not need to be too formal) (**Hint:** look what happens for "small" t).

(2) What is the Lie algebra $\mathfrak{g}$ of G?

(3) (**extra credit**, +10pt) Compute e^{tA} explicitely for every element $A \in \mathfrak{g}$, and make a drawing of the one–parameter subgroups in the (x, y)–plane.