math123, Abstract Algebra II

Exam 1

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Problem 1 (15pt)

Let V be a real vector space, with a given basis $\mathcal{B} = [v_1, v_2, \ldots, v_n]$. Recall that to every vector $v \in V$, you can associate the coordinate vector $X \in \mathbb{R}^n$, defined by $v = \mathcal{B}X$. Let $< \, , \, >: \; V \times V \longrightarrow \mathbb{R}$ be a bilinear form on V.

(1) Show how to write the matrix A of $< \, , \, >$ with respect to the basis $\mathcal{B}$.
(2) Consider a change of basis $\mathcal{B}' = \mathcal{B}P$. Write the matrix A' of $< \, , \, >$ in the new basis $\mathcal{B}'$ in terms of A and P.
(3) (**extra credit**, +5pt) Prove that the change of basis formula you gave for (2) is correct.

Solution:

(1) $A = (< v_i, v_j >)_{i,j=1\cdots n}$, or $< \mathcal{B}X, \mathcal{B}Y >= X^T AY$
(2) $A' = P^T AP$
(3) $X^T A'Y \; = < \mathcal{B}'X, \mathcal{B}'Y > = < \mathcal{B}PX, \mathcal{B}PY > = \; (PX)^T A(PY) \; = X^T(P^T AP)Y$. This implies the change of basis formula.

Problem 2 (18pt)

Let $< \, , \, >$ be a bilinear form on a real vector space V of dimension 3. Suppose in a given basis $\mathcal{B} = [v_1, v_2, v_3]$ of V, the matrix A of the bilinear form $< \, , \, >$ is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let $\mathcal{B}'$ be a new basis of V, and let A' be the matrix of $< \, , \, >$ in basis $\mathcal{B}'$. Which of the following statements are necessarily true and which are not (you do NOT need to prove or justify your answers):

(1) A' has eigenvalues $1, 2, -1$.
(2) $\exists P \in O_3(\mathbb{R})$ such that $PA'P^T$ is diagonal.
(3) A' has one negative eigenvalue.

Solution:

(1) False
(2) True (since A is symmetric)
(3) True (since the signature of $< \, , \, >$ is (2,1)).

Problem 3 (15pt)

Let $A,\ B$ be positive definite hermitian matrices. Determine which of the following matrices are positive definite hermitian matrices (you do NOT need to prove or justify your answers):

(1) A^2,
(2) A^{-1},
(3) AB,
(4) $A + B$.

Solution:

(1) Yes
(2) Yes
(3) No
(4) Yes

Problem 4 (17pt)

1. State Sylvester's law for real symmetric $n \times n$ matrices.
2. Rephrase it in terms of the orbits of the action of the group $GL_n(\mathbb{R})$ on the space S of all real symmetric $n \times n$ matrices given by:

$$\begin{array}{ccccc} GL_n(\mathbb{R}) & \times & S & \longrightarrow & S \\ P\ , & & A & \mapsto & PAP^T \end{array}$$

Solution:

(1) $\forall A \in Mat_{n\times n}(\mathbb{R})$ symmetric, $\exists P \in GL_n(\mathbb{R})$ such that:

$$PAP^T = \begin{bmatrix} \mathbb{1}_p & & 0 \\ & \mathbb{1}_q & \\ 0 & & 0 \end{bmatrix},$$

where $\mathbb{1}_p$ denotes the $p \times p$ identity matrix. The pair (p, q), called signature, is uniquely determined by A.

(2) The orbits of the $GL_n(\mathbb{R})$ action on S are:

$$C_{(p,q)} = \{A \in S \mid \text{ signature } = (p, q)\}\ .$$

Problem 5 (15pt)

Let A be any real symmetric matrix. Prove that e^A is real, symmetric and positive definite.

Solution:

Obviously e^A is a real matrix, and moreover e^A is symmetric, since $(e^A)^T = e^{A^T} = e^A$. To show that it is positive definite, notice that since A is symmetric, then it is diagonalizable, with real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. But then the eigenvalues of e^A are $e^{\lambda_1}, \ldots, e^{\lambda_n} > 0$. This implies that e^A is positive definite.

Problem 6 (15pt)

Let G be the linear group

$$G \;=\; \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \;,\quad x>0, y\in\mathbb{R} \right\} \;\subset\; GL_2(\mathbb{R})$$

(1) Determine all the 2×2 real matrices A such that $e^{tA}\in G,\ \forall t\in\mathbb{R}$ (You do not need to be too formal) (**Hint:** look what happens for "small" t).
(2) What is the Lie algebra $\mathfrak{g}$ of G?
(3) (**extra credit**, +10pt) Compute e^{tA} explicitely for every element $A\in\mathfrak{g}$, and make a drawing of the one–parameter subgroups in the (x,y)–plane.

Solution:

(1) If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $e^{tA}=\mathbb{1}+tA+O(t^2)=\begin{bmatrix} 1+ta & tb \\ tc & 1+td \end{bmatrix}+O(t^2)$. So if we impose $e^{tA}\in G$ mod (t^2), we get: $c=d=0,\ a,b\in\mathbb{R}$. Since we have seen in class that A "infinitesimally tangent" implies that $e^{tA}\in G,\forall t$, we conclude that:

$$e^{tA}\in G\ \forall t\in\mathbb{R}\ \text{ iff } A=\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\ ,\quad a,b\in\mathbb{R}\ .$$

(2) The Lie algebra $\mathfrak{g}$ of G is nothing but the space of matrices A such that $e^{tA}\in G\ \forall t$. Therefore:

$$\mathfrak{g} \;=\; \left\{ A=\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\ ,\quad a,b\in\mathbb{R} \right\}\ .$$

(3) Given $A\in\mathfrak{g}$ we can compute its powers:

$$A=\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\ ,\quad A^2=aA\ ,\qquad \text{thus } A^n=a^{n-1}A\ \forall n\geq 1\ .$$

We can then easily compute the exponential:

$$\begin{aligned} e^{tA} &= \sum_{n\geq 0}\frac{1}{n!}(tA)^n \;=\; \mathbb{1}+\sum_{n\geq 1}\frac{1}{n!}t^n a^{n-1}A \\ &= \mathbb{1}-\frac{1}{a}A+\frac{1}{a}\left(\sum_{n\geq 0}\frac{1}{n!}(ta)^n\right)A \;=\; \mathbb{1}-\frac{1}{a}A+\frac{1}{a}e^{ta}A \\ &= \mathbb{1}+(e^{ta}-1)\frac{1}{a}A \;=\; \begin{bmatrix} e^{ta} & \frac{b}{a}(e^{ta}-1) \\ 0 & 1 \end{bmatrix}\ . \end{aligned}$$

In other words, the equations defining the one parameter subgroup $\{e^{tA},\ t\in\mathbb{R}\}$ are:

$$x=e^{ta}>0\ ,\ y=\frac{b}{a}(x-1)\ .$$

Namely they are the (semi) lines (with $x>0$), going through the point $(x=1,y=0)$.