# 123 Solution Set 1

**7.1.1** Let $X_i$ be the vector with a 1 in the $i$th position and zeros elsewhere. Then, if we write $A = (a_{ij})$ and $B = (b_{ij})$, we have $a_{ij} = X_i^t A X_j = X_i^t B X_j = b_{ij}$, so we're done.
**7.1.4** First note that $(2^{-1/2}, 0)$ has norm 1 with respect to the given bilinear form. We now apply Gram-Schmidt starting from the vector $(0,1)$. We have that

$$(0,1) - ((2^{-1/2},0)^t A(0,1)) \cdot (2^{-1/2},0) = (0,1) - 2^{-1/2}(2^{-1/2},0) = (-1/2,1)$$

is orthogonal to $(2^{-1/2},0)$, and we can easily check that its norm with respect to this bilinear form is $3/2$. Therefore, an orthonormal basis is given by

$$\{(2^{-1/2},0),(-6^{-1/2},(2/3)^{1/2})\}.$$

**7.2.5** As many of you pointed out, the normality condition is false for a general symmetric bilinear form. Therefore, to be precise, we assume Artin is talking about the standard dot product. The condition that the columns of $A$ are orthonormal is equivalent to the statement that $A^tA = I$, which implies in particular that both $A$ and $A^t$ are invertible. We have

$$A^tA = 1 \iff A = (A^t)^{-1} \iff I = (A^t)^{-1}A^{-1} \iff I = (AA^t)^{-1} \iff AA^t = 1$$

which is equivalent to the statement that the rows of $A$ are orthonormal.
**7.2.7** The eigenvalues of the matrix $A$ associated to ba symmetric bilinear form are *dependent* on the choice of basis. To see this, note that the matrix for the standard dot product with respect to the basis $\{(1,0),(0,1)\}$ is $I$, whereas with respect to the basis $\{(2^{-1/2},0),(0,1)\}$ it is $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
**7.2.9** The norm $< 1, 1 >= 2$, so we take $2^{-1/2}$ as our first orthonormal basis vector. Using Gram-Schmidt, we can take for the second basis vector

$$x - \left(\int_{-1}^{1} 2^{-1/2} x dx\right) \cdot 2^{-1/2} = x.$$

We calculate $< x, x >= 2/3$, so we take as our second orthonormal basis vector $(3/2)^{1/2}x$. Again applying Gram-Schmidt, we can take as our third basis vector

$$\begin{aligned} & x^2 - \left(\int_{-1}^{1} x^2((3/2)^{1/2}x)dx\right) \cdot (3/2)^{1/2}x - \left(\int_{-1}^{1} x^2(2^{-1/2})dx\right) \cdot 2^{-1/2} \\ = \;& x^2 - 1/3. \end{aligned}$$

We calculate $< x^2 - 1/3, x^2 - 1/3 >= 8/45$, so we take $(3/2)(5/2)^{1/2}(x^2 - 1/3)$ for our third orthonormal basis vector. Finally, for the fourth, we again use Gram-Schmidt:

$$\begin{aligned} x^3 &- \left(\int_{-1}^{1} x^3((3/2)(5/2)^{1/2}(x^2-1/3))dx\right)\cdot(3/2)(5/2)^{1/2}(x^2-1/3) \\ &- \left(\int_{-1}^{1} x^3(3/2)^{1/2}x dx\right)\cdot(3/2)^{1/2}x - \left(\int_{-1}^{1} x^3 2^{-1/2}dx\right)\cdot 2^{-1/2} \\ &= x^3 - \left(\int_{-1}^{1} x^3(3/2)^{1/2}x dx\right)\cdot(3/2)^{1/2}x \\ &= x^3-(3/2)x. \end{aligned}$$

Now we compute $< x^3-(3/2)x, x^3-(3/2)x >= 41/70$, and so we may take as our final orthonormal basis (which restricts to solve each subproblem):

$$\left\{2^{-1/2}, \left(\frac{3}{2}\right)^{1/2}x, \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)^{1/2}\left(x^2-\frac{1}{3}\right), \left(\frac{70}{41}\right)^{1/2}\left(x^3-\frac{3x}{2}\right)\right\}.$$

**7.2.10** It's evident by the additivity of the trace that this is a bilinear form. We check that it is positive definite. Note that trace$(A^tA)$ is the sum of the standard $\mathbb{R}^n$ norm on the column vectors of $A$, which is certainly greater than or equal to zero. Further, it follows that this sum is zero if and only if the $\mathbb{R}^n$ norm of every column vector is zero, which means that every entry in that vector is zero. Thus $\langle A, A\rangle = 0$ if and only if $A$ is identically zero.

We now write down an orthogonal basis. We define $E_{ij}$ to be the matrix with 1 in the $i$th row and $j$th column, and zeros elsewhere. It's easy to check that the set of $E_{ij}$ as $i, j$ run from 1 to $n$ form an orthonormal basis, especially if one thinks of trace$(A^tB)$ as just being the sum of the dot products of the $j$th column of $A$ with the $j$th column of $B$.

**7.2.18** Consider the sets $D := \{E_{ii}\}_i$, $P := \{E_{ij}+E_{ji}\}_{i<j}$ and $M := \{E_{ij}-E_{ji}\}_{i<j}$. It's easy to see that the union of these three sets forms a basis for the space of $n\times n$ matrices over $\mathbb{R}$ that is orthogonal with respect to the given form $\langle,\rangle$. Further, $\langle d, d\rangle = \langle p, p\rangle = 1$ for $d\in D, p\in P$, and $\langle m, m\rangle = -1$ for $m\in M$. The sets $P$ and $M$ each have $n$ choose 2 elements, and the set $D$ has $n$. Thus it follows that the signature is $\left(\frac{n^2+n}{2}, \frac{n^2-n}{2}, 0\right)$.

**7.4.2** Consider $(i, 0, \cdots, 0)\cdot(i, 0, \cdots, 0) = -1$.

**7.4.5** Suppose $A, B$ are hermitian, and $c_1, c_2 \in \mathbb{R}$. We have

$$(c_1A+c_2B)^* = \overline{c_1}A^* + \overline{c_2}B^* = c_1A^* + c_1B^* = c_1A + c_2B$$

using the standard properties of complex numbers under conjugation. Since the zero matrix is evidently hermitian, it follows that the set of hermitian matrices is a real vector space.

By the symmetry relation $A^* = A$ for hermitian matrices, it is easy to see that any upper triangular matrix with entries over $\mathbb{C}$ and real entries on the diagonal can be completed in a unique manner to a hermitian matrix by changing the entries below the diagonal. With this in mind, we give the following basis, with $E_{jk}$ defined as above:

$$\{E_{jj}\}_j \cup \{E_{jk} + E_{kj}\}_{j<k} \cup \{iE_{jk} - iE_{kj}\}_{j<k}.$$

It's pretty evident that this set spans the space and is linearly independent.
**7.4.16.a** First note that

$$\int_0^{2\pi} \overline{e^{im\theta}} e^{in\theta} d\theta = \int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta = 2\pi\delta_{m,n}, \tag{1}$$

where $\delta_{n,m}$ is the typical Kronecker delta function. It's clear that the given form is indeed bilinear and hermitian. We check that it is positive definite. Write $f(x) = \sum_{j=0}^{n} a_j x^j$ for some $a_j \in \mathbb{C}$. Then

$$\begin{aligned} \int_0^{2\pi} \overline{f(e^{i\theta})} f(e^{i\theta}) d\theta &= \sum_{j=1}^{n} \overline{a_j} a_j \int_0^{2\pi} e^{-ij\theta} e^{ij\theta} d\theta + \sum_{j\neq k} \overline{a_j} a_k \int_0^{2\pi} e^{-ij\theta} e^{ik\theta} d\theta \\ &= 2\pi \sum_{j=1}^{n} \overline{a_j} a_j \geq 0 \end{aligned}$$

using (1), with equality if and only if all of the coefficients $a_j$ are zero.
**7.4.16.b** This is straightforward from (1). We may take our orthonormal basis to be

$$\left\{ \frac{1}{\sqrt{2\pi}} x^n \right\}_{n=0}^{1}.$$

**7.5.4** Take

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2}i & -2^{-1/2}i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2^{-1/2} & -2^{-1/2}i \\ 2^{-1/2} & 2^{-1/2}i \end{pmatrix}.$$

One checks

$$\begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2}i & -2^{-1/2}i \end{pmatrix} \begin{pmatrix} 2^{-1/2} & -2^{-1/2}i \\ 2^{-1/2} & 2^{-1/2}i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Determine the maximal value of $k \in \mathbb{Z}_{\geq 0}$ such that there exists a collection of vectors $S = \{X_1, ..., X_k\} \subset \mathbb{R}^n$ with the property that $X_1 \cdot X_j < 0$ for all $i \neq j$.**

We claim that $k = n+1$. We first note (though it is, strictly speaking, unnecessary for our argument) that we can take all of the vectors to be unit vectors, for multiplying a vector by a positive scalar does not change whether or not that vector has negative dot product with any other vector, that is, if $c \in \mathbb{R}_{>0}$, $X \cdot Y < 0$ if and only if $(cX) \cdot Y = cX \cdot Y < 0$. We first show that $k \leq n+1$ by induction. The $n = 1$ case is clear; we just take our two vectors to be $\pm 1$; any third vector in $\mathbb{R}$ must positive or negative, which means it has positive dot product with one of these vectors.

Now suppose we have the statement that we can only have $n$ vectors with the desired property in $\mathbb{R}^{n-1}$. Suppose we had a set of $k$ vectors with the desired property in $\mathbb{R}^n$. Replacing these vectors by their image under a rotation if necessary (which will not change the quantities $X_i \cdot X_j$), we may assume that one of them is the standard basis vector $e_1$. If $v_2, ..., v_k$ are the other $k-1$ vectors, note that $e_1 \cdot v_i < 0$ implies that the first coordinate of $v_i$ with respect to the standard basis must be negative. This implies that, if we write $v_i = (v_{i1}, ..., v_{in})$ with respect to the standard basis, then $v_{i1}v_{j1} > 0$. Thus $\sum_{k=2}^{n} v_{ik}v_{jk}$ must be strictly negative for $i \neq j$. In other words, if we project the $v_i$ onto the vectors in $\mathbb{R}^{n-1}$ defined by their last $n-1$ coordinates, these vectors form a set with the desired property (ie $X_i \cdot X_j < 0$ for all $i \neq j$). By the induction hypothesis, we thus have $k \leq n$, which finishes the inductive argument when we add in $e_1$.

To show that $k \geq n+1$, if $e_1, ..., e_n$ denote the standard basis elements of $\mathbb{R}^n$, then consider the set of vectors

$$\{e_1, -\delta e_1 + e_2, -\delta e_1 - \delta e_2 + e_3, ..., -\delta e_1 - \delta e_2 - ... - \delta e_{n-1} + \delta e_n, -\delta e_1 - ... - \delta e_n + e_{n+1}\}$$

with $\delta > 0$ chosen so that $n\delta^2 < \delta$. It's easy to see that this set of $n+1$ vectors satisfies our requirements.