$\boxed{5.2.12}$

(a) We proved the first part in 5.1.8.

Prove $\underset{T}{\overset{?}{E_\lambda}} = \underset{T^{-1}}{\overset{?}{E_{\lambda^{-1}}}}$

By defn, $E_\lambda = \{x \in V \mid T(x) = \lambda x\} = N(T - \lambda I_V)$.

But $T(x) = \lambda x \iff T^{-1}(y) = \lambda^{-1}(y)$, where $y = \lambda x$ $(\Rightarrow x = \lambda^{-1} y)$

So there's a one-to-one correspondence b/w the two eigenspaces.

(Alternatively, you can show that there's an isomorphism b/w them).

(b) T diagonalizable $\Rightarrow T^{-1}$ diagonalizable.

By (a), the same eigenvectors that are in the eigenspace of T are in the eigenspace of T^{-1}. These are linearly independent $\Rightarrow$ we can find a basis for the eigenspace of T^{-1} consisting of all its eigenvectors $\Rightarrow T^{-1}$ diagonalizable.

$\boxed{5.2.13}$ E_λ = eigenspace of A

E'_λ = " " A^t

(a) Example (Thanks to Barbara Richter) - There are many, many others you can find.

$A = \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}$, $A^t = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ ~ both have eigenvalues $\lambda_1 = 2, \lambda_2 = -2$

but: $E_{\lambda_2} = \text{Span}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $E'_{\lambda_2} = \text{Span}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\times$

(b) $A \leftrightarrow L_A$, $A^t \leftrightarrow L_{A^t}$

By defn, $E_\lambda = N(L_A - \lambda I)$

but $\dim[N(L_A - \lambda I)] + \dim[R(L_A - \lambda I)] = \overbrace{\dim[\mathbb{F}^n]}^{N} \Rightarrow$ ✓

$\Rightarrow \dim(E_\lambda) = N - \dim[R(L_A - \lambda I)]$.

By defn. $E'_\lambda = N(L_{A^t} - \lambda I) = N(L_A - \lambda I)^t = R(L_A - \lambda I)^\circ$ by previous hw. exercise

$\Rightarrow \dim(E'_\lambda) = N - \dim[R(L_A - \lambda I)]$. ✓

(c) A diagonalizable $\Rightarrow A^t$ diagonalizable.

Let $\{\lambda_i\}$ be the eigenvalues of A, $m_i = \text{mult.}(\lambda_i)$ $(i = \overline{1,k})$.

$\Rightarrow \{\lambda_i^*\}$ are the eigenvalues of A^t.

For $\forall \lambda_i$, we have: $\dim(E_{\lambda_i}) = m_i$

by (b), $\dim(E'_{\lambda_i}) = m_i$, $\forall \lambda_i$ eigenvalue of A^t

Since we also know that char poly of both A & A^t split (by hypothesis)

$\Rightarrow A^t$ is diagonalizable

$\boxed{\text{Q.E.D.}}$