So: $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ ordered basis

and $[T]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\boxed{5.1.8}$

(a) A linear operator T on a finite-dim'l vector space is invertible $\Leftrightarrow$ zero is not an eigenvalue of T.

"$\Rightarrow$" T invertible $\Rightarrow$ $N(T) = \{0\}$ necessarily.
If $\lambda = 0$ were an eigenvalue of T, we'd have: $T(v) = 0$ for some $v \neq 0$
this is a contradiction!

"$\Leftarrow$" zero is not an eigenvalue of T $\Rightarrow$ $N(T) = \{0\}$ (by contradiction)
$\Rightarrow$ T is one-to-one.
Since $T : V \to V$, $\dim(V) =$ finite $\Rightarrow$ T is also onto by Thm 2.5
Hence T is bijective, and thus invertible.

(b) λ is an eigenvalue of $T \Leftrightarrow \lambda^{-1}$ is an eigenvalue of T^{-1}.
for T invertible.

From (a) & T invertible, we know both λ and λ^{-1} exist.
This is one of the few "If and only if" proofs that you can do in one step:

λ eigenvalue of T $\Leftrightarrow$ $Tv = \lambda v$ $\Leftrightarrow$ (left-mult. by T^{-1})
$\Leftrightarrow T^{-1}Tv = T^{-1}\lambda v \Leftrightarrow$
$\Leftrightarrow v = T^{-1}(\lambda v) \Leftrightarrow$ Let $u = \lambda v \Rightarrow v = \lambda^{-1}u$. (defined)
$\Leftrightarrow \lambda^{-1}u = T^{-1}(u) \Leftrightarrow$
$\Leftrightarrow \lambda^{-1}$ is an eigenvalue of T^{-1}

(c) (a) Matrix $A \in M_{n\times n}(F)$ invertible $\Leftrightarrow$ zero is not an eigenvalue of A
The proof follows immediately from the association of A with a linear op.
(b) If ~~matrix~~ $A \in M_{n\times n}(F)$ is invertible, then λ is an eigenvalue of $A \Leftrightarrow \lambda^{-1}$ is an eigenvalue of A^{-1}.
Once again, the proof is identical to the one for linear operators.