So $\exists\, Q, S$ invertible matrices s.t. $Q^{-1}AQ = J = S^{-1}A^t S$

$\Rightarrow \quad A = (QS^{-1})A^t(SQ^{-1}) \Rightarrow$

$\Rightarrow \quad A = T^{-1}A^t T$

Hence A, A^T are similar matrices. Q.E.D.

$\boxed{7.28}$ (Thanks to Barbara Richler)

ⓐ Take $\beta = \{x_1, \ldots, x_n\}$, $[T]_\beta = \begin{pmatrix} \begin{matrix}\lambda_1 & 1 & \\ & \ddots & 1 \\ & & \lambda_1\end{matrix} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \begin{matrix}\lambda_k & 1 & 0 \\ & \ddots & 1 \\ & & \lambda_k\end{matrix} \end{pmatrix}$

β is a Jordan canonical basis, so we either have:

$T(x_i) = \lambda x_i$ or $T(x_i) = x_{i-1} + \lambda x_i$ for some eigenvalue λ.

Consider $\beta' = \{v_1, \ldots, v_n\}$, $v_j = c x_j$, $j = \overline{1,n}$.

$$\Rightarrow \begin{cases} \underline{T(v_i)} = T(cx_i) = c\,T(x_i) = c(\lambda x_i) = \lambda(cx_i) = \underline{\lambda v_i} & \text{or:} \\ \underline{T(v_i)} = T(cx_i) = c\,T(x_i) = c(x_{i-1} + \lambda x_i) = cx_{i-1} + \lambda(cx_i) = \\ \qquad = \underline{v_{i-1} + \lambda v_i} \end{cases}$$

So β' is also a Jordan canonical basis for T.

ⓑ $\gamma = \{(T-\lambda I)^k(x), \ldots, (T-\lambda I)(x), x\}$.

$\gamma' = \{(T-\lambda I)^k(x+y), \ldots, (T-\lambda I)(x+y), x+y\}$.

First prove γ' is cycle of gen. eigenvect. corresp to λ:

$\left.\begin{array}{l} y \in N(T-\lambda I) \subseteq N((T-\lambda I)^{k+1}) \\ x \in N((T-\lambda I)^{k+1}) \end{array}\right| \Rightarrow x+y \in N((T-\lambda I)^{k+1})$.

Now prove that replacing γ by γ' in defn of β, we end up w/ Jordan canonical basis:

$(T-\lambda I)(x+y) = (T-\lambda I)(x) + (T-\lambda I)y = (T-\lambda I)(x)$ (← why is this?)

So: $\gamma' = \{(T-\lambda I)^k(x), \ldots, (T-\lambda I)(x), x+y\}$

Know: $T(x) = (T-\lambda I)x + \lambda x$. So just check: $T(x+y) = (T-\lambda I)x + \lambda(x+y)$ which works, so we are done!

ⓒ Eg. $\beta' = \left\{ \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$.

or $\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$