

1 Section 6.1

1.1 Problem 8

1.1.1 (a)

This is not an inner product because $\langle (1, 1), (1, 1) \rangle = 0$

1.1.2 (b)

This is not an inner product because $\langle A, A \rangle = 0$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1.1.3 (c)

This is not an inner product because $\langle c, c \rangle = 0$ where c is any constant polynomial.

1.2 Problem 11

$$|x + y|^2 + |x - y|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle = 2 \langle x, x \rangle + 2 \langle x, y \rangle - 2 \langle x, y \rangle + 2 \langle y, y \rangle = 2|x|^2 + 2|y|^2$$

The lengths of the diagonals of a parallelogram in \mathbb{R}^2 are $|x + y|$ and $|x - y|$, so this says that the squares of the lengths of the diagonals is equal to twice the sum of the lengths of the sides.

1.3 Problem 12

$|\sum_{i=1}^k a_i v_i|^2 = \langle \sum_{i=1}^k a_i v_i, \sum_{i=1}^k a_i v_i \rangle = \sum_{1 \leq i, j \leq k} a_i a_j \langle v_i, v_j \rangle$ by linearity. Since the set is orthogonal this equals $\sum_{i=1}^k |a_i|^2 |v_i|^2$.

2 Section 6.2

2.1 Problem 6

By Theorem 6.6, we can write x uniquely as $y + w$ with $y \in W^\perp, y \neq 0$ and $w \in W$. Since $x \notin W$ and $y \in W^\perp$ we have $\langle x, y \rangle \neq 0$.

2.2 Problem 10

By Theorem 6.6, we can write $v = u + w \forall v \in V$ uniquely, where $w \in W, u \in W^\perp$. Define $T : V \rightarrow V$ by $T(u + w) = w$. Then T is the required projection by definition, and $N(T)$ is all vectors $v = u + w$ with $w = 0$, i.e. $v = u \in W^\perp$. So $N(T) = W^\perp$.

Using the notation above, $|T(v)|^2 = |w|^2 \leq |w|^2 + |u|^2 = |w + u|^2 = |v|^2$. So $|T(v)| \leq |v|$.

2.3 Problem 11

Let $A = [A_1, A_2, \dots, A_n]^t$, i.e. the A_i are row vectors which are the rows of A . Let the A_i be an orthonormal basis for \mathbb{C}^n . Then $(AA^*)_{ij} = \langle A_i, A_j^* \rangle$ by definition of matrix multiplication, and since the A_i are an orthonormal basis, $(AA^*)_{ij} = \delta_{ij}$ i.e. $AA^* = I$.

Let $AA^* = I$. Then $\langle A_i, A_j^* \rangle = \delta_{ij}$ as above, which shows that the A_i are an orthonormal basis.

2.4 Problem 13

2.4.1 (a)

Let $s \in S^\perp$. Then s is perpendicular to every vector in S and $S_0 \subset S$, so $s \in S_0^\perp$. Therefore $S^\perp \subset S_0^\perp$.

2.4.2 (b)

if $s \in S$, then for any $v \in S^\perp$, $\langle v, s \rangle = 0$, so $s \in (S^\perp)^\perp$

2.4.3 (c)

all we need is $(W^\perp)^\perp \subset W$. By exercise 6, if $x \notin W$ then $x \notin (W^\perp)^\perp$ so the contrapositive is $x \in (W^\perp)^\perp \Rightarrow x \in W$.

2.4.4 (d)

We showed this in class - all you need to check is that $V = W + W^\perp$ and $W \cap W^\perp = 0$.

2.5 Problem 14

$\langle v, w_1 + w_2 \rangle = 0 \forall w_i \in W_i \Rightarrow \langle v, w_i \rangle = 0$ for any $w_i \in W_i$ (let $w_j = 0$). $\langle v, w_i \rangle = 0 \forall w_i \in W_i \Rightarrow \langle v, w_1 + w_2 \rangle = 0$ by linearity. Hence $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$. For the second part, note $(W_1^\perp + W_2^\perp)^\perp = W_1 \cap W_2$ from above and applying $^\perp$ to each side gives the desired result.

2.6 Problem 16

2.6.1 (a)

Using the notation of Theorem 6.6 and the problem, write $x = u + z$ where $u \in W = \text{span}(S)$. Then $|x|^2 \geq |u|^2 = \left| \sum_{i=1}^n \langle y, v_i \rangle v_i \right|^2 = \sum_{i=1}^n |\langle x, v_i \rangle|^2$ by linearity and orthogonality.

2.6.2 (b)

Note that the equality in part (a) holds iff $x = u \in \text{span}(S)$.

2.7 Problem 18

Let $f \in W_e^\perp$. Then for any even g , $0 = \langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$ so by splitting up the integral and using the fact that g is even, $\int_0^1 f(-t)g(t)dt = \int_0^1 f(t)g(t)dt \Rightarrow f \in W_o$. If $f \in W_o$ then we split up the integral into the same two parts, and using the fact that f is odd and g is even we therefore have fg is odd, and so $\langle f, g \rangle = 0$. Therefore $W_e^\perp = W_o$.