28 March 2004 Name (PRINT): _____

Math 121 : Practice second midterm

Instructions

- Please print your name at the top of each page of this exam.
- This exam consists of 5 questions. Some questions have several parts. You should answer all parts of all questions.
- Write your answers on the exam paper. Continue onto the back of the page if necessary and ask for more paper if you need it.
- You may use any results which were proved in class, as part of the first midterm, or in the textbook provided that you state them clearly. You *may not* use results which were proved as part of your homework unless you prove them again here.
- Unless otherwise indicated, all vector spaces should be taken to be over the field $\mathbb{C}$ of complex numbers.

((Please do not write beneath this line))

1	/ 20
2	/ 15
3	/ 10
4	/ 15
5	/ 10
Total:	/ 70

(1) For each of the following statements, give either a short proof (if the statement is true) or an example or short argument which shows that the statement is false.

(a) The matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 7 \\ -2 & -8 & 1 \end{pmatrix}$$

are similar. *(5 points)*

(b) If a matrix A is obtained by finitely many row operations from a matrix B then the rank of A is equal to the rank of B. *(5 points)*

((Question 1 continues on the next page))

(c) The nullity of the transformation

$$\begin{aligned} T : \mathsf{P}_4(\mathbb{R}) &\to \mathsf{P}_3(\mathbb{R}) \\ f(x) &\mapsto f''(x) + 2f'(x) \end{aligned}$$

is 2. *(5 points)*

(d) The transformation

$$\begin{aligned} T : \mathsf{P}_2(\mathbb{R}) &\to \mathsf{P}_2(\mathbb{R}) \\ f(x) &\mapsto f(x) + f(1)x \end{aligned}$$

is diagonalizable. *(5 points)*

(2) For this question, work over the field $\mathbb{R}$ of real numbers.

(a) Let V be a vector space and $T, U : V \to V$ be linear maps. Suppose that $N(T)$ and $N(U)$ are finite-dimensional and that U is onto. Show that

$$\dim(N(TU)) = \dim(N(T)) + \dim(N(U))$$

Use this to show that the solution space to the differential equation

$$y'' - 4y = 0$$

is 2-dimensional. *(11 points)*

((Question 2 continues on the next page))

(b) Show that the set of solutions to the differential equation

$$y'' - 4y = -2$$

is

$$\{\cos^2 t + ae^{2t} + be^{-2t} : a, b \in \mathbb{R}\}$$

(4 points)

(3) (a) Suppose that V is a finite-dimensional vector space and that W is a subspace of V. Show that

$$\dim(V/W) = \dim V - \dim W$$

(5 points)

(b) Let $T : V \to Z$ be a linear map. Show that the induced map

$$\overline{T} : V/N(T) \to R(T)$$

is an isomorphism. *(5 points)*

(4) Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear map.
   (a) Show that $R(T)^0 = N(T^t)$. *(5 points)*
   (b) Write down necessary and sufficient conditions for T to be diagonalizable. *(2 points)*
   (c) Show that T is diagonalizable if and only if T^t is diagonalizable. *(8 points)*

(5) (a) Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear map. Suppose that $v_1, \ldots, v_k$ are eigenvectors of T with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that the set $\{v_1, \ldots, v_k\}$ is linearly independent. *(4 points)*

(b) Let V be an n-dimensional vector space and let $T : V \to V$ be a linear map. Suppose that T has exactly two eigenvalues, λ_1 and λ_2, and that the dimensions of the corresponding eigenspaces are k and $n - k$ respectively. Show that T is diagonalizable. *(6 points)*

((This is the last question on the exam))