PRACTICE MIDTERM

Instructions

Answer all questions. You may use any results which were proved in class or in the textbook provided that you state them clearly. You *may not* use results which were proved as part of your homework unless you prove them again here.

Question 1 (5 points per part)

For each of the following statements, give either a short proof (if the statement is true) or a counterexample (if the statement is false):

- if $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ are linear maps such that $UT : \mathbb{R}^2 \to \mathbb{R}^2$ is the zero transformation then either U is the zero transformation or T is the zero transformation.
- if β_1 and β_2 are subsets of a vector space V then
$$\operatorname{span}(\beta_1) \cap \operatorname{span}(\beta_2) \subseteq \operatorname{span}(\beta_1 \cap \beta_2)$$
- if V, W and Z are subspaces of $\mathbb{R}^3$ and $V \oplus W = V \oplus Z$ then $W = Z$.
- the map $T : P(\mathbb{R}) \to P(\mathbb{R})$ which sends a polynomial $p(x)$ to $p(x^2)$ is linear.

Question 2 (10 points)

Consider the map

$$\begin{aligned} T : P_2(\mathbb{R}) &\to P_3(\mathbb{R}) \\ f &\mapsto \int f(x)\, dx \end{aligned}$$

Let β and γ be the standard bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$ respectively.

- Compute $[T]_\beta^\gamma$.
- What is the rank of T?

Question 3 (15 points)

- State and prove the Dimension Theorem.
- Let
$$\begin{aligned} W_0 &= \{f \in P_n(\mathbb{R}) : f(1) = 0\} \\ W_1 &= \{f \in P_n(\mathbb{R}) : f(1) = 1\} \end{aligned}$$
Show that W_0 is a subspace of $P_n(\mathbb{R})$ and that W_1 is not.
- Find the dimension of W_0.

Question 4 (5 points each part)

Let V be the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. Let

$$\begin{aligned} g_1 : V &\rightarrow \mathbb{R} \\ f &\mapsto f(1) \end{aligned}$$

$$\begin{aligned} g_2 : V &\rightarrow \mathbb{R} \\ f &\mapsto f(2) \end{aligned}$$

$$\begin{aligned} h : V &\rightarrow \mathbb{R} \\ f &\mapsto \int_0^1 f(x)\,dx \end{aligned}$$

- Show that g_1, g_2 and h are all elements of $V^\star$.
- Show that the set $\{g_1, g_2\}$ is linearly independent.
- Is h in the span of $\{g_1, g_2\}$?

Question 5 (10 points)

A linear map $T : V \rightarrow V$ from a vector space V to itself is called *left-invertible* if and only if there exists a map $S : V \rightarrow V$ such that $ST = I_V$. A linear map $T : V \rightarrow V$ from a vector space V to itself is called *right-invertible* if and only if there exists a map $S : V \rightarrow V$ such that $TS = I_V$.

- Suppose that V is finite-dimensional. Show that $T : V \rightarrow V$ is left-invertible if and only if it is right invertible. (To put this another way: show that T is left-invertible if and only if it is invertible.)
- Give an example of an infinite-dimensional vector space V and a linear transformation $T : V \rightarrow V$ which is right-invertible but not left-invertible.