12 May 2004 Name (PRINT): ______________________

Math 121 : Practice final

Instructions

- Please print your name at the top of each page of this exam.
- This exam consists of 7 questions. Some questions have several parts. You should answer all parts of all questions.
- Write your answers on the exam paper. Continue onto the back of the page if necessary and ask for more paper if you need it.
- Unless otherwise indicated, you may use any results which were proved in class, as part of the first or second midterm, or in the textbook provided that you state them clearly. You *may not* use results which were proved as part of your homework unless you prove them again here.
- Unless otherwise indicated, all vector spaces should be taken to be over the field $\mathbb{C}$ of complex numbers.

((Please do not write beneath this line))

1	/ 20
2	/ 10
3	/ 15
4	/ 8
5	/ 17
6	/ 18
7	/ 12
Total:	/ 100

(1) (a) The *trace* $\mathrm{tr}(A)$ of a square matrix A is the sum of the diagonal entries of A. Show that if A and B are $n \times n$ matrices then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

Deduce that similar matrices have the same trace. *(5 points)*

(b) Show that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j)$$

(Hint: to show that the determinant is divisible by $x_i - x_j$, show that it vanishes when $x_i - x_j = 0$.) *(5 points)*

((Question 1 continues on the next page))

(c) Define inner products on the vector spaces $\mathsf{P}_2(\mathbb{R})$ and $\mathsf{P}_1(\mathbb{R})$ by setting

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

Let

$$\begin{aligned} T : \mathsf{P}_2(\mathbb{R}) &\rightarrow \mathsf{P}_1(\mathbb{R}) \\ f &\mapsto f'' \end{aligned}$$

Find the rank r of T, the non-zero singular values $\sigma_1 \geq \ldots \geq \sigma_r$ of T, and orthonormal bases $\{f_1, f_2, f_3\}$ for $\mathsf{P}_2(\mathbb{R})$ and $\{g_1, g_2\}$ for $\mathsf{P}_1(\mathbb{R})$ such that $T(f_i) = \sigma_i g_i$.

(5 points)

(d) Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by rotation by 45 degrees about the axis which passes through the origin and $(1, 1, 1)$, rotating clockwise as you look from the origin towards $(1, 1, 1)$. Is T self-adjoint? Is T orthogonal? *(5 points)*

(2) State and prove the Dimension Theorem. *(10 points)*

(3) (a) Let $V = \mathsf{P}_1(\mathbb{R})$ and define $f_1,\ f_2 \in V^\star$ by

$$f_1(p(x)) = \int_0^1 p(t)\,dt \qquad \text{and} \qquad f_2(p(x)) = \int_0^2 p(t)\,dt$$

Show that $\{f_1, f_2\}$ is a basis for $V^\star$, and find a basis for V for which it is the dual basis. *(8 points)*

(b) Let W be a vector space over $\mathbb{C}$ and let $f, g : W \to \mathbb{C}$ be linear maps such that $f(w) = 0$ if and only if $g(w) = 0$. Show that there exists $\alpha \in \mathbb{C}$ such that $f = \alpha g$. *(7 points)*

(4) (a) Let a and b be real. Show that a basis for the subspace of $C^\infty(\mathbb{R})$ consisting of solutions $y(t)$ to the differential equation

$$y'' - 2ay' + (a^2 + b^2)y$$

is

$$\{e^{at}\cos(bt), e^{at}\sin(bt)\}$$

(4 points)

(b) The displacement from rest $y(t)$ of a mass hanging from the end of a spring satisfies the above differential equation, where $a = -1$ and $b = 3$. Suppose that at time $t = 0$ the mass is in its rest position and the velocity y' is equal to v. Find the displacement $y(t)$ at time t. What happens to the oscillation of the mass as $t \to \infty$?

(4 points)

(5) Let V be a vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation.

(a) Suppose that β and γ are bases for V such that

$$A = [T]_\beta^\beta \quad \text{and} \quad B = [T]_\gamma^\gamma$$

are both in Jordan canonical form. Give a careful explanation of why A and B are "nearly the same", *i.e.* why they differ only by a reordering of their Jordan blocks. *(8 points)*

(b) Consider the transformation

$$\begin{aligned} T : \mathsf{P}_3(\mathbb{R}) &\to \mathsf{P}_3(\mathbb{R}) \\ f &\mapsto f'' + 2f \end{aligned}$$

Find a basis β for $\mathsf{P}_3(\mathbb{R})$ such that the matrix $C = [T]_\beta^\beta$ is in Jordan canonical form. Also, find C. *(7 points)*

(c) Find another basis δ for $\mathsf{P}_3(\mathbb{R})$ such that $\delta \neq \beta$ but $[T]_\delta^\delta = [T]_\beta^\beta$. *(2 points)*

(6) Let V be a complex inner product space. A linear transformation $T : V \to V$ is called *normal* if and only if it commutes with its adjoint, *i.e.*

$$TT^\star = T^\star T$$

(a) Show that if R and S are linear transformations from V to V then

$$(R + \lambda S)^\star = R^\star + \bar{\lambda} S^\star$$

(3 points)

(b) Show that if $T : V \to V$ is normal then

$$\|T(x)\| = \|T^\star(x)\|$$

for all $x \in V$. *(2 points)*

(c) Show that if $T : V \to V$ is normal and $x \in V$ is an eigenvector for T with eigenvalue λ then x is an eigenvalue for $T^\star$ with eigenvalue $\bar{\lambda}$.
(Hint: x is an eigenvector for T with eigenvalue λ if and only if $(T - \lambda I)x = 0$.)
(3 points)

(d) Show that if $T : V \to V$ is normal then there is an orthonormal basis for V consisting of eigenvectors for T.
(Hint: think about how we proved the corresponding result for self-adjoint operators.)
(8 points)

(e) Give an example of a complex vector space V and a linear transformation $T : V \to V$ which is normal but not self-adjoint. *(2 points)*

(7) Let V be a vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation.

(a) Show that T is nilpotent if and only if all eigenvalues of T are zero. *(5 points)*

(b) The *trace* $\mathrm{tr}(T)$ of T is defined to be $\mathrm{tr}(A)$, where A is the matrix of T with respect to some basis β. Show that this is well-defined, *i.e.* that this does not depend on the choice of basis β. *(2 points)*

(c) Show that T is nilpotent if and only if all of $\mathrm{tr}(T)$, $\mathrm{tr}(T^2)$, $\ldots$, $\mathrm{tr}(T^n)$ are zero. *(5 points)*

((This is the last question on the exam))