first part of the question, we see that

$$\text{dim of solution space} = \dim N(D^2 - 4I) = 2$$

(b) Claim: The solution space to $y'' - 4y = 0$ is $\{ae^{2t} + be^{-2t} : a, b \in \mathbb{R}\}$

Proof: $\left(\frac{d}{dt}\right)^2 (e^{2t}) - 4e^{2t} = 0$

and $\left(\frac{d}{dt}\right)^2 (e^{-2t}) - 4e^{-2t} = 0$

so $y = e^{2t}$ and $y = e^{-2t}$ are both in the solution space. I claim that they are linearly independent. If not, then

$$\alpha e^{2t} + \beta e^{-2t} = 0$$

for some $\alpha, \beta \in \mathbb{R}$ not both zero (and hence neither α nor β is zero). Letting $t \to \infty$ we see that α must be zero $\times$

Thus $\{e^{2t}, e^{-2t}\}$ is a basis for the solution space to $y'' - 4y = 0$. □

Claim: $y = \cos^2 t$ solves $y'' - 4y = -2$

Proof $y'' = \frac{d}{dt}(2 \sin t \cos t)$
$= 2\cos^2 t - 2\sin^2 t$. Thus $y'' - 4y = -2$. □