$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

But rank T = rank $[T]_\beta^\gamma$ which is, using the fact that column operations are rank preserving:

$$\text{rank} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 4$$

So the rank of T is 4.

(d) Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$.
Then

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the characteristic polynomial of T is

$$f(t) = \begin{vmatrix} 1-t & 0 & 0 \\ 1 & 2-t & 1 \\ 0 & 0 & 1-t \end{vmatrix} = \begin{vmatrix} 1-t & 0 & 0 \\ 0 & 2-t & 0 \\ 0 & 0 & 1-t \end{vmatrix}$$

$$= (1-t)^2(2-t)$$

The eigenspace corresponding to eigenvalue 1 is

$$N\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$