SOLUTIONS TO SECOND PRACTICE MIDTERM

1 (a) The matrices are not similar : similar matrices have the same determinant, but

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 24 \quad \text{and} \quad \begin{vmatrix} 1 & 4 & 2 \\ 0 & 5 & 7 \\ -2 & -8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 2 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 25.$$

(b) If B is obtained from A by a sequence of k row operations then $B = E_k E_{k-1} \cdots E_1 A$ where $E_1, \ldots, E_k$ are elementary matrices.
But elementary matrices are invertible, and left-multiplying by an invertible matrix does not change the rank of a matrix, so $\operatorname{rank}(B) = \operatorname{rank}(A)$.

To prove that, if A is an $n \times k$ matrix and E is an invertible $n \times n$ matrix then $\operatorname{rank} EA = \operatorname{rank} A$, one could observe that E is invertible $\Leftrightarrow$ E^T is invertible, and

$$\begin{aligned} \operatorname{rank}(EA) = \operatorname{rank}(EA)^T &= \operatorname{rank} A^T E^T \\ &= \dim \{A^T E^T x : x \in \mathbb{R}^n\} \\ &= \dim \{A^T y : y \in \mathbb{R}^n\} \\ &= \operatorname{rank} A^T \\ &= \operatorname{rank} A \end{aligned}$$

(c) Let $\beta = \{1, x, x^2, x^3, x+1\}$ be the standard basis for $P_4(\mathbb{R})$ and $\gamma = \{1, x, x^2, x^3\}$ be the standard basis for $P_3(\mathbb{R})$. Then (PTO)