

Q5

(a)

 T invertible $\Rightarrow T$ left invertible

(9)

If T is invertible then $\exists S: V \rightarrow V$ with

$$ST = I_V \quad \text{and} \quad TS = I_V.$$

In particular, $ST = I_V$ so T is left-invertible T left-invertible $\Rightarrow T$ invertibleIf T is left-invertible then $\exists S: V \rightarrow V$ with

$$ST = I_V.$$

I need to show that ① T is one-to-one② T is ontoBy the Dimension Theorem, as ~~through~~ T goes from V to V we have ① \Rightarrow ②.So I just need to show that T is one-to-one.But T is one-to-one $\Leftrightarrow N(T) = \{0\}$.Suppose that T is not one-to-one.Then $N(T) \neq \{0\}$, so there is a vector x with $x \neq 0$ and $T(x) = 0$.But $ST = I_V$, so $S(T(x)) = x$

$$\Rightarrow S(0) = x$$

$$\text{but also } S(T(0)) = 0$$

$$\Rightarrow S(0) = 0$$

~~as~~ as $x \neq 0$ Thus T is one-to-one, and hence invertible.