

Q4

(a)

I need to show that g_1, g_2 and h are linear. Let f and \tilde{f} be arbitrary elements of V and let $c \in \mathbb{R}$ be an arbitrary scalar. (7)

Then:

$$\begin{aligned} (*) \quad g_1(f + \tilde{f}) &= (f + \tilde{f})(1) \\ &= f(1) + \tilde{f}(1) \\ &= g_1(f) + g_1(\tilde{f}) \end{aligned}$$

$$\begin{aligned} (*) \quad g_1(cf) &= (cf)(1) \\ &= c f(1) \\ &= c g_1(f) \end{aligned}$$

$$\begin{aligned} (*) \quad g_2(f + \tilde{f}) &= (f + \tilde{f})(2) \\ &= f(2) + \tilde{f}(2) \\ &= g_2(f) + g_2(\tilde{f}) \end{aligned}$$

$$\begin{aligned} (*) \quad g_2(cf) &= (cf)(2) \\ &= c f(2) \\ &= c g_2(f) \end{aligned}$$

$$\begin{aligned} (*) \quad h(f + \tilde{f}) &= \int_0^1 (f + \tilde{f})(x) dx \\ &= \int_0^1 f(x) + \tilde{f}(x) dx \\ &= \int_0^1 f(x) dx + \int_0^1 \tilde{f}(x) dx \\ &= h(f) + h(\tilde{f}) \end{aligned}$$

$$\begin{aligned} (*) \quad h(cf) &= \int_0^1 cf(x) dx \\ &= c \int_0^1 f(x) dx = c h(f) \end{aligned}$$